Mechanics and Oscillations

University Physics I: Notes and exercises

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UNIVERSITY PHYSICS I: NOTES AND EXERCISES

Mechanics and Oscillations: University Physics I: Notes and exercises
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CONTENTS

1 INTRODUCTION TO MECHANICS

Your goal for this chapter is to learn about measurement, significant figures and coordinate systems.

1.1 MEASUREMENT

Measurement is comparison with a standard. For example when we say the length of a certain object is *3* m we are saying the length of the object is *3* times the length of the standard meter. The standard with which the comparison is made is called a unit of measurement. For example the unit of measurement for length is the meter

There are two systems of units. These are the British System and the SI system. SI is an abbreviation for the French phrase 'Systeme Internationale'. The British system is used in the United States while the SI system is used in most of the rest of the world. For scientific purposes, the SI system is used. In this course, the SI system will be used.

The SI System of Units: The SI system of units are the units based on standards kept in an SI office in France. These units may be classified into two: Fundamental units and derived units.

Fundamental Units: These are the minimum set of units from which all the units of physics can be assembled. The standards for SI fundamental units are kept in the SI office in France. Manufacturers of fundamental units should base their units on these standards. The following table is a list of the fundamental units of physics.

Physical Quantity	Unit	Abbreviation
Length	meter	m
Time	second	s
Mass	kilogram	kg
Temperature	degree Kelvin	°K
Current	Ampere	A

The units of length (meter), time (second) and mass (kilogram) are the fundamental unit of Mechanics. The unit of temperature (°K) is the fundamental unit of thermodynamics (study of heat). The unit of current (A) is the fundamental unit of electricity and magnetism.

Derived Units: Derived units are units that can be expressed as a combination of fundamental units. For example the unit of speed is a derived unit because it can be expressed as a ratio between the unit of length and the unit of time (m / s). The following table is a list of some derived units of physics.

Physical Quantity	Unit	Abbreviation
Speed	meter/second	m/s
Acceleration	meter/second^2	m/s^2
Volume	meter^3	m^3
Density	kilogram/meter^3	kg/m^3

Common Abbreviations of Powers of Ten: The units of physics are defined in such a way that they can be comprehended by human senses. For example the kilogram is a weight we can hold in our hand; the second is an interval of time we can comprehend and the meter is about twice of human arm. But in physics, quite often we deal with quantities which are either much bigger or much smaller than these units. To deal with such quantities conveniently, names and abbreviations for some powers of ten are defined. For example the "kilo" and "k" are the name and the abbreviation for a *1000*. The following table is a list of some commonly used powers of ten.

Power of Ten	Name	Abbreviation
10^3	kilo	k
10^6	Mega	M
10^9	Gega	G
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	µ
10^{-9}	nano	n

1.2 SIGNIFICANT FIGURES

Significant figures are digits of a report of a measurement that make sense. For example let say the length of a certain object is measured by a ruler whose least count is a cm and the measurement is reported as *4.321* cm. Using this device, it is possible to determine that the length of the object is between *4* cm and *5* cm. The ones digit (*4*) can be determined exactly. The tenth digit (*3*) can be determined approximately because we know the length is between *4* cm and *5* cm. Since we are not sure of the tenth digit (*3*), it is impossible to determine the hundredth digit (*2*) and the thousandth digit (*1*). Thus, we say the ones digit (*4*) and the tenth digit (*3*) are significant digits while the hundredth digit (*2*) and the thousandth digit (*1*) are not significant digits.

Scientifically, significant digits include all accurate digits and one uncertain digit. A report of a measurement should include only significant digits. For example the length of the object in our example should be reported as *4.3* cm. But sometimes we will be forced to include zeroes that are not significant digits because zeroes are used to hold decimal places. For example let say the length of an object is measured by a device whose least count is *100* cm and its length is found to be between *200* cm and *300* cm. The tens digit can be approximated (let say it is *5*) but the ones digit cannot be determined. Even though the ones digit cannot be determined we have to put zero in its place in order to indicate that the digit *2* is a hundreds digit. The measurement is reported as *250* cm. It is important that we are able to tell whether a zero included in a report of a measurement is significant or not

Determining whether Zeroes in a Report of Measurement are Significant or not: We can determine whether a zero included in a report of a measurement is significant or not by following the following rules.

1. A non-zero digit in a report of a measurement is always significant.
2. Tailing zeroes before the decimal point are not significant. They are used to hold decimal places only. For example the zeroes in *200* are not significant.
3. Tailing zeroes after the decimal point are significant. If they are not significant, they don't have to be included. For example the zeroes in *2.1000* are significant.
4. Leading zeroes are not significant. They are used to hold decimal places only. For example the zeroes in *0.0023* are not significant.
5. Zeroes located between significant digits are significant. This is because the zero holds a higher decimal place than another significant digit. For example the zeroes in 3001 are significant.

Example: How many significant digits are there in the following reports of a measurement?

a) *3000*
Solution: one, because the zeroes are tailing zeroes before the decimal point.

b) *2001*
Solution: four, because the zeroes are located between significant digits

c) *0.00453*
Solution: three, because the zeroes are leading zeroes

d) *100.0*
Solution: four, because the last zero is a tailing zero after the decimal point and the other zeroes are located between significant digits

Operating with Significant Figures: When operating (adding, subtracting, multiplying, dividing) with significant figures, the result cannot be more accurate (have greater number of significant digits) than either of the figures being operated.

Adding or Subtracting Significant Figures: The result of adding or subtracting significant figures should have the same number of decimal places as the figure with the least number of decimal places. For example when adding *2.13* and *3.4571*, the sum should have only two decimal places because one of the figure (*2.13*) has two decimal places (*1* and *3*) and the second number (*3.4571*) has four decimal places (4, 5, 7 and 1). Even though the algebraic addition of the figures gives *5.5871*, to obtain significant figures this should be rounded to two decimal places and the result should be reported as *5.59*.

Multiplying or Dividing Significant Figures: The result of multiplying or dividing significant numbers should have the same number of significant digits as the figure with the least number of significant digits. For example when multiplying *200* by *38*, the result should have only one significant digit because one of the figures (*200*) has only one significant digit and the other figure (*38*) has two significant digits. Even though direct multiplication gives *7600*, to obtain significant figures, this should be rounded to ten thousands decimal place and the result should be reported as *8000*.

Standard (Scientific) Notation: Expressing a number in standard notation means expressing a number as a product between a number between one (inclusive) and ten and a power of ten. It allows you to express a report of a measurement as a product of a number that consists of significant digits only and a power of ten. The non-significant zeroes are absorbed in the power of ten. For example to express *2400* in standard notation, first we have to divide it by 1000 to get a number between one (inclusive) and ten. This gives *2.4*. And then of course we have to multiply by a *1000* or 10^3 to represent the original number (*2400*). Thus the standard notation of *2400* is 2.4×10^3. Similarly, to express *0.0540* in standard notation first we multiply it by *100* to change it to a number between one(inclusive) and ten which gives *5.40* and then multiply it by *10* to the power 0f *-2*. Thus its standard notation is 5.40×10^{-2}. The zero is included in *5.40* because it is significant.

Adding or Subtracting Numbers in Standard Notation: To add or subtract numbers in standard notation, first manipulate the numbers so that all of them have the same powers of ten. Then, factor out the power of ten and operate. For example to add the numbers 2×10^2 and 3×10^3, first change the power of ten of the first number to *3* by dividing the *2* by *10* and multiplying the power of ten by *10*. This gives 0.2×10^3. Then factor out the power of ten to get $(0.2 + 3) \times 10^3$. And the result is 3.2×10^3.

Multiplying or Dividing Numbers in Standard Notation: To multiply or divide numbers in standard notation, multiply (divide) numbers with numbers and powers of ten with powers of ten. For example to multiply 2×10^2 and 3×10^3, multiply the numbers (*2* and *3*) together and the powers of ten (10^2 and 10^3) together to get 6×10^5.

Practice Quiz 1.1

Choose the best answer

1. The System of units used for scientific purposes is the
   A) SI system
   B) British system
   C) European system
   D) American system

2. *2* nm means
   A) *2e9* m
   B) *2e-6* m
   C) *2e-9* m
   D) *2e6* m

3. Fundamentals units of mechanics are units of
   A) mass, volume, density
   B) length, mass, and time
   C) length, time, temperature
   D) distance, speed, and acceleration

4. The measurement of the length of a certain rod is reported as 12.34 cm. Which of these digit(s) is (are) uncertain?
   A) 2, 3 and 4
   B) all of the digits are accurate
   C) 4
   D) 3 and 4

5. How many significant digits are there in the significant figure 0.000756
   A) 6
   B) 3
   C) 1
   D) 7

6. How many significant digits are there in the significant figure 2003
   A) 2
   B) 1
   C) 4
   D) 3

7. How many significant digits are there in the significant figure 24.00
   A) 1
   B) 2
   C) 4
   D) 3

8. The number of significant digits in the significant figures 0.000500, 600100, and 0.0004020 respectively are
   A) 1, 6, and 3
   B) 3, 4, and 4
   C) 1, 4, and 4
   D) 3, 4, and 2

9. The sum of the significant figures 56, 2.15, and 0.5643 gives the significant figure
   A) 58.7143
   B) 58
   C) 59
   D) 58.7

10. The product of the significant figures 1.34 and 2100 gives the significant figure
    A) 2800
    B) 3000
    C) 2810
    D) 2814

11. Express *560000* in standard notation
    A) *5.6e4*
    B) *56e4*
    C) *5.6e5*
    D) *5.6e-5*

12. Express *0.5* in standard notation
    A) *0.5*
    B) *5e-1*
    C) *5e1*
    D) *0.5e0*

1.3 CONVERSION OF UNITS

To convert from one unit to another, first find a relationship between the units. Then use this relationship to convert from one to the other. A relationship between the units may be obtained by finding the ratio between the two units.

Example: Convert *5* km to mm.

Solution: First we have to find a relationship between km and mm by finding the ratio between km and mm. The unit meter (m) cancels out, so the ratio is basically ratio between kilo (k) and milli (m). Remember k = 10^3 and m = 10^{-3}.

$$\text{Km}/\text{mm} = \text{k}/\text{m} = 10^3/10^{-3} = 10^6$$

$$\text{km} = 10^6 \text{ mm}$$

$$5 \text{ km} = 5 \times 10^6 \text{ mm}$$

Example: Convert $5\ \mathrm{cm}^2$ to m^2

Solution: First we have to find the ratio between the two units. The unit (m^2) will cancel out and the ratio simplifies to c^2 (The square in cm applies to both c and m). Remember $c = 10^{-2}$.

$$\mathrm{cm}^2 / \mathrm{m}^2 = c^2 = 10^{-4}$$

$$\mathrm{cm}^2 = 10^{-4}\ \mathrm{m}^2$$

$$5\ \mathrm{cm}^2 = 5 \times 10^{-4}\ \mathrm{m}^2$$

1.4 DIMENSIONAL ANALYSIS

Dimensional Analysis is a method used to determine if an equation is sound or not by comparing the units of both sides of the equation. Both sides of an equation are obviously expected to have the same units. If they turn out to be different, then the equation must be wrong; if they turn out to be the same, then the equation is at least dimensionally correct and there is a good chance it may be correct.

Example: Determine if the equation *speed* = *acceleration x time* is dimensionally correct.

Solution: The unit for speed is m/s. The unit for acceleration is m/s^2. Thus, the unit for the left side of the equation is m/s and that of the right hand side is the product of m/s^2 and s which is equal to m/s. Since both sides have the same units, the equation is dimensionally correct.

1.5 ORDER OF MAGNITUDE CALCULATION

Calculating the order of magnitude of a certain calculation means approximating its power of ten. To obtain the order of magnitude, replace every number by the power of ten closest to it and then carry out the calculation.

Example: Obtain the order of magnitude of the following calculation: $34 * 4 * 786 * 9876$

Solution: *34* is between 10^1 and 10^2 and is closer to 10^1. *4* is between 10^0 and 10^1 and is closer to the former. *786* is between 10^2 and 10^3 and is closer to the later. *9876* is between 10^3 and 10^4 and is closer to the later. Therefore the order of magnitude of this calculation is

$$10^1 * 10^0 * 10^3 * 10^4 = 10^8$$

1.6 BRIEF REVIEW OF TRIGONOMETRY

Trigonometric Functions: There are three basic trigonometric functions which are the cosine, the sine and the tangent. They are defined in terms of a right angled triangle. A right angled triangle is a triangle whose largest angle's measure is *90°*. Its longest side or the side opposite to the *90* degree angle is called the hypotenuse. The other two sides are called the legs of the right angled triangle. Now consider one of the none 90° angles. This angle is formed by the hypotenuse and one of the legs. The latter is called the adjacent side. The other leg which is not part of this angle is called the opposite side. Let this angle be x. the adjacent side be a, the opposite side be b and the hypotenuse be c.

The cosine of this angle, written as $\cos x$, is defined to be the ratio between the adjacent side and the hypotenuse.

$$\cos x = a/c$$

The sine of this angle, written as $\sin x$, is defined to be the ratio between the opposite side and the hypotenuse.

$$\sin x = b/c$$

The tangent of this angle, written as $\tan x$, is defined to be the ratio between the opposite side and the opposite side.

$$\tan x = b/a$$

The values of trigonometric functions are available in a scientific calculator.

The hypotenuse and the legs of a right angled triangle are related by Pythagorean Theorem.

$$c^2 = a^2 + b^2$$

Example: The degree measure of one of the angles of a right angled triangle is 60°. The hypotenuse is *10*. Calculate its adjacent side.

Solution: $c = 10$; $x = 60°$; $a = ?$

The trigonometric function that relates these 3 values is cosine.

$$\cos x = a/c$$

$$a = c \cos x = 10 \cos 60° = 5$$

Inverse Trigonometric Functions: The inverses of trigonometric functions, called inverse trigonometric functions, help you to recover an angle from the value of a trigonometric function.

Cosine inverse of a trigonometric value a, written as $\cos^{-1} a$ or $\arccos a$, is defined as follows:

If

$$\cos x = a,$$

then

$$x = \arccos a$$

Sine inverse of a trigonometric value a, written as $\sin^{-1} a$ or $\arcsin a$, is defined as follows:

If

$$\sin x = a,$$

then

$$x = \arcsin a$$

Tangent inverse of a trigonometric value *a*, written as $tan^{-1}\ a$ or *arctan a*, is defined as follows:

If

$$tan\ x = a,$$

then

$$x = arctan\ a$$

Example: In a certain right angled triangle of hypotenuse *50*, the side opposite to one of the angles, *x*, is *25*. Calculate the angle.

Solution: $c = 50;\ b = 25;\ x = ?$

$$sin\ x = b/c = 25/50 = 0.5$$

$$x = arcsin\ 0.5 = 30°$$

Trigonometric Identities: The following is a list of some common trigonometric identities.

$$tan\ x = (sin\ x)/(cos\ x)$$

$$\cos^2 x + \sin^2 x = 1$$

$$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$$

$$\cos(x \pm y) = \cos(x)\cos(y) \pm (-)\sin(x)\sin(y)$$

$$\sin(2x) = 2\sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

1.7 COORDINATE SYSTEMS

A coordinate system is a system for associating a set of three numbers with points in space uniquely. Two special cases of this are the one dimensional coordinate system and the two dimensional coordinate system.

One Dimensional Coordinate System: A one dimensional coordinate system is also called a number line. It is a system that associates single numbers with points on a line uniquely. A point is related with a number that is equal to its distance from a certain point that we call a reference point or origin. To distinguish between distances to the right of the origin and distances to the left of the origin, distances to the right of the origin are taken to be positive while distances to the left of the origin are taken to be negative.

Two Dimensional Coordinate System: A Two dimensional coordinate system is also called a coordinate plane. It is a system for associating pairs of numbers with points in a plane uniquely. There are two kinds of two dimensional coordinate system which are the Cartesian coordinate system and the polar coordinate system.

The Cartesian coordinate system: In this kind of coordinate system, a point is related with its perpendicular distances from two reference lines that are perpendicular to each other. The vertical reference line is called the y-axis and the horizontal reference line is called the x-axis (They don't have necessarily to be vertical and horizontal. But the horizontal-vertical coordinate system is the most common one). The intersection point of this two lines is called the origin. The perpendicular distance between the point and the y-axis is called the x-coordinate of the point. Distances to the right of the y-axis are taken to be positive while distances to the left of the y-axis are taken to be negative. The perpendicular distance between the point and the x-axis is called the y-coordinate of the point. y-coordinates for points above the x-axis are taken to be positive and y-coordinates for points below the x-axis are taken to be negative. The x-coordinate and y-coordinate of a point are customarily represented by the letters x, and y respectively. The coordinate of the point is represented by the ordered pair (x, y).

The Polar coordinate system: In this kind of coordinate system, a point is related to its distance from the origin and the angle formed between the line joining the origin to the point and the positive x-axis. The angle is taken to be positive if measured in a counter clockwise direction from the positive x-axis and negative if measured in a clockwise direction from the positive x-axis. The distance and the angle are customarily represented by r and θ respectively. The coordinate of the point is represented by the ordered pair (r, θ).

Relationship between Cartesian and Polar Coordinates: Consider the right angled triangle formed by the side joining the origin to the given point and a line extended from the point to the x-axis perpendicularly. The length of the hypotenuse is equal to r. The angle formed between the horizontal leg and the hypotenuse is equal to θ. The lengths of the horizontal and vertical legs are respectively equal to the x and y coordinates of the point (if in the first quadrant). The following relationships between Cartesian and polar coordinates can be obtained easily using the definitions of trigonometric functions.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{(x^2 + y^2)}$$

$$\theta = \arctan (y/x)$$

Example: The polar coordinate of a certain point is $(100, 60°)$. Calculate its Cartesian coordinates.

Solution: $r = 100$; $\theta = 60°$; $x = ?$; $y = ?$

$$x = r \cos \theta = 100 \cos 60° = 50$$

$$y = r \sin \theta = 100 \sin 60° = 87$$

Example: The Cartesian coordinate of a certain point is $(64, 48)$. Calculate its polar coordinates.

Solution: $x = 64$; $y = 36$; $r = ?$; $\theta = ?$

$$r = \sqrt{(x^2 + y^2)} = \sqrt{(64^2 + 48^2)} = 80$$

$$\theta = \arctan (y/x) = \arctan (48/64) = 37°$$

Practice Quiz 1.2

Choose the best answer

1. Use dimensional analysis to determine which of the following equations is dimensionally correct
   A) $acceleration * distance = time / speed$
   B) $acceleration * distance = speed^2 / time$
   C) $acceleration * distance = speed / time^2$
   D) $acceleration * distance = speed^2$
   E) $acceleration * distance = speed / time$

2. Approximate the order of magnitude of the following product: $346 * 8 * 5632 * 25 * 3 * 980$
   A) *1e11*
   B) *1e10*
   C) *1e13*
   D) *1145819136000*
   E) *1e9*

3. Convert *3* mm is equal to
   A) *3e-6* km
   B) *3e-3* km
   C) *3e6* km
   D) *3e-5* km
   E) *3e5* km

4. *6* cm / min is equal to
   A) *0.001* m / s
   B) *0.1* m / s
   C) *360* m / s
   D) *3600* m / s
   E) *0.006* m / s

5. *4* cm^2 is equal to
   A) *4e-1* m^2
   B) *4e-4* m^2
   C) *4e2* m^2
   D) *4e-2* m^2
   E) *4e4* m^2

6. *9* kg / m^3 is equal to
   A) *90,000* g / cm^3
   B) *9000* g / cm^3
   C) *0.09* g / cm^3
   D) *0.0009* g / cm^3
   E) *0.009* g / cm^3

7. The hypotenuse of a right angled triangle is 12 m long. The length of one of the legs is 9. Calculate the length of the other leg
   A) *3.571* m
   B) *5.033* m
   C) *7.937* m
   D) *5.934* m
   E) *9.182* m

8. The hypotenuse of a right angled is *10* m. Calculate the length of one of the legs if the angle between this leg and the hypotenuse is *50* degrees.
   A) *5.687* m
   B) *8.164* m
   C) *9.788* m
   D) *6.428* m
   E) *11.67* m

9. The hypotenuse of a right angled triangle is *10* m long. Calculate the length of one of its legs, if the angle opposite to this leg is *60* degrees.
   A) *16.272* m
   B) *14.688* m
   C) *8.66* m
   D) *2.681* m
   E) *1.255* m

10. One of the legs of a right angled triangle is *30* m long. The angle between this leg and the hypotenuse is *80* degrees. Calculate the length of the other leg.
   A) *220.723* m
   B) *170.138* m
   C) *271.096* m
   D) *29.447* m
   E) *130.836* m

11. The hypotenuse of a right angled triangle is *12* m long. One of its legs is *5* m long. Calculate the angle formed between this leg and the hypotenuse.
   A) *22.792* deg
   B) *97.713* deg
   C) *42.668* deg
   D) *58.341* deg
   E) *65.376* deg

12. The Cartesian coordinates of a certain point are $(x, y) = (35, 40)$ m. Calculate the polar coordinates (r, θ).
   A) (*53.151* m, *75.446* deg)
   B) (*30.667* m, *48.814* deg)
   C) (*53.151* m, *48.814* deg)
   D) (*6.851* m, *55.061* deg)
   E) (*30.667* m, *75.446* deg)

13. The polar coordinates of a certain point are $(r, \theta) = (72 \text{ m}, 20 \text{ deg})$. Calculate its Cartesian coordinates (x, y).s)
    A) $(46.554 \text{ m}, 24.625 \text{ m})$
    B) $(46.554 \text{ m}, 10.826 \text{ m})$
    C) $(67.658 \text{ m}, 24.625 \text{ m})$
    D) $(67.658 \text{ m}, 10.826 \text{ m})$

2 MOTION IN ONE DIMENSION

You goal for this chapter is to learn the relationships among motion variables for motion in a straight line.

Motion in one dimension is motion in a straight line. Let's start with a brief review of calculus.

2.1 BRIEF REVIEW OF CALCULUS

Calculus is a branch of mathematics that studies relationships between changes systematically. To be exact, it deals with rate of change of one variable with respect to another variable. If y is a function of x, then the average rate of change of y with respect to x is equal to the ratio between the change in y and the change in x; that is, if (x_i, y_i) is the initial point and (x_f, y_f) is the final point, then the average rate of change of y with respect to x is equal to $\frac{\Delta y}{\Delta x} = \frac{y_f - y_i}{x_f - x_i}$. Graphically, the average rate of change of y with respect to x is equal to the slope of the straight line joining the initial point and the final point

The instantaneous rate of change of y with respect to x (rate of change of y with respect to x at a given point), which is represented as $\frac{dy}{dx}$, is equal to the limit of the average rate of change of y with respect to x as Δx approaches zero.

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

This is also called the derivative of y with respect to x. The purpose of the first part of calculus is basically to express the derivative $\left(\frac{dy}{dx}\right)$ as a function of x provided y is known as a function of x. For example if $y = x^n$ where n is a constant, it can be shown that

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

The following rules are useful when dealing with combinations of functions. If f and g are arbitrary functions of x, and c is a constant then,

$$\frac{dc}{dx} = 0$$

$$\frac{d}{dx}\left[cf\right] = c\frac{df}{dx}$$

$$\frac{d}{dx}[f \pm g] = \frac{df}{dx} \pm \frac{dg}{dx}$$
$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$
$$\frac{d}{dx}f(g(x)) = \frac{df(g)}{dg}\frac{dg(x)}{dx}$$

Example: Find the derivative of $y = 4x^3$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left[4x^3\right] = 4\frac{dx^3}{dx} = 4\left(3x^{3-1}\right) = 12x^2$$

Example: Find the derivative of $y = 2x^4 - 3x^2 + 5$.

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}\left[2x^4 - 3x^2 + 5\right] = \frac{d}{dx}\left[2x^4\right] + \frac{d}{dx}\left[3x^2\right] + \frac{d}{dx}[5]$$
$$= 2\frac{d}{dx}\left(x^4\right) - 3\frac{d}{dx}\left(x^2\right) + \frac{d[5]}{dx} = 2\times 4x^3 - 3\times 2x^1 + 0 = 8x^3 - 6x$$
$$= 2\times 4x^3 - 3\times 2x^1 + 0 = 8x^3 - 6x$$

Graphically, the derivative of y with respect to x at a given point can be obtained from the graph of y versus x as the slope of the line tangent to the curve at the given point.

The second part of calculus deals with the inverse of a derivative. Its purpose is to express y as a function of x, provided $\frac{dy}{dx}$ as known as a function of x. This inverse process is called integration. The symbol of integration is '$\int$'.

$$\text{If } \frac{dy}{dx} = f(x), \text{ then } y(x) = \int f(x)dx + c$$

The arbitrary constant c is called constant of integration. It is needed because all functions of the form $y(x) + c$ have the same derivative (because the derivative of c is zero). Because of this, this kind of integral is called indefinite integral. An additional condition is needed to determine c. For example, since $\frac{dx^{n+1}}{dx} = (n+1)x^n$, it follows that $\int (n+1)x^n dx = x^{n+1} + c$ or

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

The following are useful rules of combinations. If f and g are arbitrary functions of x, and c is a constant, then

$$\int c\, f(x)\,dx = c\int f(x)dx$$

$$\int\left[f(x)\pm g(x)\right]dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Example: Find the integral of

Solution:

$$\int\left(3x^3 - 4x + 2\right)dx = \int 3x^3 dx - \int 4x\,dx + 2\int dx + c$$

$$= 3\int x^3\,dx + 4\int x\,dx + 2\int dx + c = \frac{3x^4}{4} - \frac{4x^2}{2} + 2x + c$$

To understand the graphical meaning of the integral, consider the area enclosed between the graph of $f(x)$ versus x curve and the x-axis. Let $A(x)$ be the area enclosed between the $x = a$ and an arbitrary value x and let $A(x+\Delta x)$ be the area enclosed between $x = a$ and $x+\Delta x$. Then $\Delta A = A(x+\Delta x) - A(x)$ is equal to the area of a rectangle of base Δx and height $f(x)$; that is; $\Delta A = f(x)\Delta x$. If Δx is taken to be infinitesimally small, then $f(x) = \lim\limits_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \frac{dA}{dx}$ which implies that $A(x) = \int f(x)dx + c$. Therefore the integral of a function $f(x)$ is equal to the area enclosed between the $f(x)$ versus x curve and the x-axis. The constant c is required because $x = a$ is an arbitrary constant. The area enclosed between two known values of x, say x_i *and* x_f is unique (without arbitrary constant) and is called definite integral. The definite integral between $x = x_i$ and $x = x_f$ is written as $\int\limits_{x=x_i}^{x=x_f} f(x)dx$.

$$\text{If } \int f(x)dx = F(x) + c, \text{ then } \int\limits_{x=x_i}^{x=x_f} f(x)dx = F(x_f) - F(x_i)$$

Example: Calculate the area enclosed between $x = 2, x = 4$ and $y = x^2$.

Solution:

$$\int\limits_{x=2}^{x=4} x^2 = \frac{x^3}{3}\Big|_{x=4} - \frac{x^3}{3}\Big|_{x=2} = \frac{64}{8} - \frac{8}{3} = \frac{56}{3}$$

2.2 MOTION VARIABLES

Position(x)*:* is a physical quantity used to represent the location of a particle. Specification of position requires a reference point or origin. Positions to the right of the origin are taken to be positive while those to the left are taken to be negative. (For vertical motion up is positive and down is negative). SI unit of position is meter (m).

Displacement (Δx)*:* is defined to be change in position.

$$\Delta x = x_f - x_i$$

Where x_i and x_f represent the initial and final positions of the particle respectively. Displacement does not depend on the choice of a reference point. Displacement to the right (left) is positive (negative). Infinitely small displacement is represented by *dx*.

Average Velocity ($\bar{v}$)*:* is defined to be displacement per unit time.

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

Average velocity to the right (left) is positive (negative). The unit of measurement for velocity is meter/second.

Instantaneous Velocity (v)*:* is velocity at a given instant of time or it is average velocity evaluated at an infinitely small interval of time.

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

Instantaneous velocity is equal to the derivative of position with respect to time.

Average acceleration ($\bar{a}$)*:* is defined to be change in velocity per a unit time.

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$$

Average acceleration is positive (negative) if velocity is increasing (decreasing). Unit of measurement for acceleration is meter/second^2 .

Instantaneous acceleration (a)*:* is acceleration at a given instant of time; or it is average acceleration evaluated at infinitely small interval of time.

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

Instantaneous acceleration is equal to the derivative of velocity with respect to time.

Since $v = \frac{dx}{dt}$, we may also write $a = \frac{d}{dt}\left(\frac{dx}{dt}\right)$. In calculus this is written as $a = \frac{d^2x}{dt^2}$; that is, acceleration is equal to the second derivative of position with respect to time.

Example: The position of a certain particle varies with time according to the equation $x = a_1t^3 - a_2t^2 + a_3t + a_4$ where $a_1 = 4\ \text{m/s}^3$, $a_2 = 2\ \text{m/s}^2$, $a_3 = 1\ \text{m/s}$; $a_4 = 1\ \text{m}$.

a) Where is the particle after 2 seconds?

Solution:

$$x\big|_{t=2\ \text{s}} = a_1(2\ \text{s})^3 + a_2(2\ \text{s})^2 + a_3(2\ \text{s}) + a_4 = 27\ \text{m}$$

b) Calculate its displacement between t = 1 s and t = 2 s.

Solution:

$$\Delta x = x_f - x_i = x|_{t=2\text{ s}} - x|_{t=1\text{ s}} = \left(a_1 t^3 - a_2 t^2 + a_3 t + a_4\right)\Big|_{t=2\text{ s}} - \left(a_1 t^3 - a_2 t^2 + a_3 t + a_4\right)\Big|_{t=1\text{ s}} = 23\text{ m}$$

c) Calculate its average velocity between $t = 1$ s and $t = 2$ s.

Solution:

$$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i} = \frac{x|_{t=2\text{ s}} - x|_{t=1\text{ s}}}{t_2 - t_1} = \frac{27-4}{(2-1)}\text{ m/s} = 23\text{ m/s}$$

d) Calculate its instantaneous velocity after 4 seconds.

Solution:

$$v = a_1 \frac{d}{dt}\left(t^3\right) - a_2 \frac{d}{dt}\left(t^2\right) + a_3 \frac{d}{dt}(t) + \frac{d}{dt}\left(a_4\right) = 3a_1 t^2 - 2a_2 t + a_3$$

$$v|_{t=4\text{ s}} = 81\text{ m/s}$$

e) Calculate its average acceleration between $t = 1$ s and $t = 3$ s.

Solution:

$$a = \frac{v|_{t=3\text{ s}} - v|_{t=1\text{ s}}}{(3-1)\text{ s}} = \frac{\left(3a_1t^2 - 2a_2t + a_3\right)\Big|_{t=3\text{ s}} - \left(3a_1t^2 - 2a_2t + a_3\right)\Big|_{t=1\text{ s}}}{2\text{ s}} = 44\text{ m/s}$$

f) Calculate its instantaneous acceleration after 10 seconds.

Solution:

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(3a_1t^2 - 2a_2t + a_3\right) = 6a_1t - 2a_2$$

$$a\big|_{t=10\text{ s}} = 236\text{ m/s}^2$$

g) At what time is its acceleration zero?

Solution:

$$6a_1t - 2a_2 = 0 \Rightarrow t = \frac{1}{6}\text{ s}$$

Example: The velocity of a particle varies with time according to the equation $v = \left(2\text{ m/s}^2\right)t - 4\text{ m/s}$. Assuming initially the particle was located at the origin, find its location after 5 seconds.

Solution:

$$v = \left(2\text{ m/s}^2\right)t - 4\text{ m/s} = \frac{dx}{dt} \Rightarrow x = \int\left(\left(2\text{ m/s}^2\right)t - 4\text{ m}\right)dt + c = t^2\text{ m}^2/\text{s}^2 - 4t\text{ m/s} + c$$

$$x\big|_{t=0} = 0 \Rightarrow c = 0$$

$$x = t^2\text{ m/s}^2 - 4t\text{ m/s} \Rightarrow x\big|_{t=5\text{ s}} = 5\text{ m}$$

Example: A particle is moving with a constant acceleration of 2 m/s^2. If its initial speed is 5 m/s and it is initially located at $x = 2$ m

a) Find its velocity after 8 seconds.

Solution:

$$a = \frac{dv}{dt} = 2\ \text{m/s}^2 \Rightarrow v = \int \left(2\ \text{m/s}^2\right) dt + c = \left(2\ \text{m/s}^2\right) t + c$$

$$v\big|_{t=0} = 5\ \text{m/s} \Rightarrow c = 5\ \text{m/s}$$

$$v = \left(2\ \text{m/s}^2\right) t + 5\ \text{m/s} \Rightarrow v\big|_{t=8\ \text{s}} = 21\ \text{m/s}$$

b) Where is the particle located after 10 seconds?

Solution:

$$v = \frac{dx}{dt} = \left(2\ \text{m/s}^2\right) t + 5\ \text{m/s} \Rightarrow x = \int \left(\left(2\ \text{m/s}^2\right) t + 5\ \text{m/s}\right) dt + c' = \left(t^2\ \text{m/s}^2\right) + 5t\ \text{m/s} + c'$$

$$x\big|_{t=0} = 2\ \text{m} \Rightarrow c' = 2\ \text{m}$$

$$x = t^2\ \text{m/s}^2 + 5t\ \text{m/s} + 2\ \text{m} \Rightarrow x\big|_{t=8\ \text{s}} = 106\ \text{m}$$

Practice Quiz 2.1

Choose the best answer

1. A physical quantity used to specify the location of a particle is called
   A) distance
   B) displacement
   C) velocity
   D) acceleration
   E) position

2. The SI unit of measurement for velocity is
   A) meter / second ^{2}
   B) meter / second
   C) meter ^{2} / second ^{2}
   D) meter * second
   E) meter ^{2} / second

3. The position of a certain particle varies with time according to the equation $x = 5.6t^2 + 3.5t$. Where is the particle after 5.3 seconds?
   A) 165.793 m
   B) 179.461 m
   C) 175.854 m
   D) 169.348 m
   E) 161.915 m

4. The position of a certain particle varies with time according to the equation $x = 3.4t^2 - 4.1t$. Calculate the displacement of the particle between $t = 8.1$ and $t = 15.1$ seconds?
   A) 801.478 m
   B) 340.247 m
   C) 461.354 m
   D) 664.923 m
   E) 523.46 m

5. The position of a certain particle varies with time according to the equation $x = 5.6t^4 - 6.2$. Calculate its velocity after 5.3 seconds.
   A) 3851.475 m/s
   B) 4997.492 m/s
   C) 567.351 m/s
   D) 3334.845 m/s
   E) 1219.384 m/s

6. The position of a certain particle varies with time according to the equation $x = 3.4t^2 - 2.2$. Calculate the average velocity of the particle between $t = 4.1$ and $t = 15.1$ seconds?
   A) 109.857 m/s
   B) 65.28 m/s
   C) 123.279 m/s
   D) 95.173 m/s
   E) 9.655 m/s

7. The position of a certain particle varies with time according to the equation $x = 3.4t^3 + 6.2t^2$. Calculate the average acceleration of the particle between $t = 10.2$ and $t = 16.6$ seconds?
   A) 358.682 m/s^2
   B) 31.847 m/s^2
   C) 285.76 m/s^2
   D) 433.503 m/s^2
   E) 137.405 m/s^2

8. The position of a certain particle varies with time according to the equation $x = 5.6t^4 + 6.2t$. Calculate its acceleration after 7.3 seconds.
   A) 3581.088 m/s^2
   B) 1233.122 m/s^2
   C) 4069.527 m/s^2
   D) 5238.449 m/s^2
   E) 2935.963 m/s^2

9. The velocity of a certain particle varies with time according to the equation $v = 7.3t^3$. Obtain an expression for its displacement as a function of time.
   A) $1.46t^4$
   B) $1.825t^5$
   C) $21.9t^2$
   D) $2.433t^4$
   E) $1.825t^4$

10. The acceleration of a certain particle varies with time according to the equation $a = 7.3t^3$. If the particle started from rest, obtain an expression for its displacement as a function of time.
    A) $0.365t^5$
    B) $1.825t^5$
    C) $21.9t^2$
    D) $1.825t^4$
    E) $0.365t^4$

2.3 UNIFORMLY ACCELERATED MOTION

Uniformly accelerated motion is motion with constant acceleration; That is. $a = \frac{dv}{dt}$ is a constant, If the initial velocity is v_i and the final velocity (velocity at arbitrary time t) is v_f, then $\int_{v_i}^{v_f} dv = a\int_0^t dt'$ and

$$v_f = v_i + at$$

Again since $v = \frac{dx}{dt}$, if the initial location of the particle is x_i and the final location (location at an arbitrary time t) is x_f, then $\int_{x_i}^{x_f} dx = v_i\int_0^t dt' + a\int_0^t t'dt'$ and

$$x_f - x_i = \Delta x = v_i t + \frac{1}{2}at^2$$

These two equations are the only independent equations of a uniformly accelerated motion; which means we can solve only for two unknowns. But other dependent equations can be obtained by manipulating these equations. $\bar{v} = \frac{\Delta x}{t} = \frac{v_i t + 1/2at^2}{t} = v_i + \frac{1}{2}at = v_i + \frac{1}{2}\left(v_f - v_i\right)$. Therefore the average velocity for a uniformly accelerated motion is given as

$$\bar{v} = \frac{v_i + v_f}{2}$$

Since $\Delta x = \bar{v}t$, displacement may also be written as

$$\Delta x = \left(\frac{v_i + v_f}{2}\right)t$$

Again, with $t = \frac{v_f - v_i}{a}$, the displacement can be expressed in terms of the initial velocity, final velocity and acceleration as $\Delta x = \left(\frac{v_i + v_f}{2}\right)\left(\frac{v_f - v_i}{a}\right)$ which is customarily written as

$$v_f^2 = v_i^2 + 2a\Delta x$$

Now putting the four equations of a uniformly accelerated motion together

$$v_f = v_i + at$$

$$\Delta x = v_i t + \frac{1}{2}at^2$$

$$\Delta x = \left(\frac{v_i + v_f}{2}\right)t$$

$$v_f^{\,2} = v_i^2 + 2a\Delta x$$

There are five variables in these equations $(v_f, v_i, t, \Delta x, a)$. Since only two equations are independent, if we know any three variables, we can solve for the other variables. In using these equations, you should be careful about signs. Δx is taken to be positive if the displacement is to the right and negative if it is to the left. $v_i\,(v_f)$ is positive if the motion is to the right and negative if the motions is to the left. a is positive if the velocity is increasing and negative if the velocity is decreasing (remember increase of a negative is a decrease). t (time) is always positive.

Example: A car changes its speed from 10 m/s to 30 m/s in 10 sec.

a) Calculate its acceleration

Solution:

$$v_i = 10 \text{ m/s};\ v_f = 30 \text{ m/s}; t = 10 \text{ s}; \quad a = ?$$

$$v_f = v_i + at$$

$$a = \frac{v_f - v_i}{t} = \frac{30-10}{10} \text{ m/s}^2 = 2 \text{ m/s}^2$$

b) Calculate the distance traveled

Solution:

$$\Delta x = \left(\frac{v_i + v_f}{2}\right)t = \left(\frac{10+30}{2}\right)10 \text{ m} = 200 \text{ m}$$

Example: A car initially moving with a speed of 40 m/s was stopped in a distance of 100 m. Calculate its acceleration.

Solution:

$$v_i = 40 \text{ m/s};\ v_f = 0 \text{ (stopped)};\ \Delta x = ?$$

$$v_f^{\,2} = v_i^2 + 2a\Delta x$$

$$(40 \text{ m/s})^2 = 2a(100 \text{ m})$$

$$a = -8 \text{ m/s}^2$$

2.4 MOTION UNDER GRAVITY

Motion under gravity is a uniformly accelerated motion. The numerical value of gravitational acceleration (g) is 9.8 m/s^2 and its effect is to decrease velocity (it decreases the speeds of objects going up and it increases the negative velocity of objects going down). Therefore its sign must be negative.

$$g = -9.8 \ \frac{\text{m}}{\text{s}^2}$$

The equations of motion under gravity can be obtained easily from the equations of a uniformly accelerated motion by replacing a by g and Δx by Δy (since it is a vertical motion).

$$v_f = v_i + gt$$

$$\Delta y = v_i t + \frac{1}{2} g t^2$$

$$\Delta y = \left(\frac{v_i + v_f}{2}\right) t$$

$$v_f^2 = v_i^2 + 2g\Delta y$$

Δy is positive if the final position is above the initial position and negative if final position is below initial position. v_i and v_f are positive for upward motion and negative for downward motion. There are four variables $(v_i, v_f, t \text{ and } \Delta y)$. If any two of the variables are known, the other two variables can be determined using these equations.

Example: An object is thrown upwards with a speed of 20 m/s.

a) How long will it take to reach its maximum height?

Solution:

$v_i = 20$ m/s; $v_f = 0$(velocity is zero at maximum height); $t = ?$

$$v_f = v_i + gt$$

$$0 = 20 \text{ m/s} + \left(-9.8 \text{ m/s}^2\right) t$$

$$t = 2.04 \text{ s}$$

b) How high will it rise?

$$\Delta y = v_i t + \frac{1}{2} g t^2 = (20 \times 2.04 + 0.5 \times -9.8 \times 2.04^2) \text{ m} = 20.4 \text{ m}$$

Example: A ball is dropped from a height of *20* m.

a) Calculate the time taken to reach the ground.
Solution: $v_i = 0$ (because it is dropped from rest); $\Delta y = -20$ m (negative, because it is going down); $t = ?$

$$\Delta y = v_i t + (g/2)t^2 = (g/2)t^2 \text{ (since } v_i = 0 \text{ m/s)}$$

$$t = \sqrt{(2\Delta y/g)} = \sqrt{2(-20)/(-9.8)} \text{ s} \approx 2 \text{ s}$$

b) Calculate its speed by the time it hits the ground.
Solution: $v_f = ?$

$$v_f = v_i + at = \{0 + (-9.8)(2)\}\ \text{m/s} \approx = -20\ \text{m/s}$$

Example: A ball is thrown upward with a speed of 20 m/s.

a) Calculate the time taken to reach the maximum height.
Solution: $v_i = 20$ m/s; $v_f = 0$ m/s (At maximum height speed is zero).

$$v_f = v_i + at$$

$$t = (v_f - v_i)/g = (0 - 20)/(-9.8) \approx 2\ \text{s}$$

b) To what height would it rise?
Solution: $\Delta y = ?$

$$\Delta y = (v_i + v_f)(t/2) = (20 + 0)(2/2)\ \text{m} \approx 20\ \text{m}$$

Example: A ball is thrown upwards from a 10 m tall building with a speed of 10 m/s.

a) Calculate the speed with which it will hit the ground.
Solution: $v_i = 10$ m/s; $\Delta y = -10$ m (negative because the final position is below the initial position); $v_f = ?$

$$v_f^2 = v_i^2 + 2g\Delta y$$

$$v_f = \sqrt{v_i^2 + 2g\Delta y} = -\sqrt{10^2 + 2(-9.8)(-10)}\ \text{m/s} \approx -17.3\ \text{m/s}$$

The negative square root is taken because it is going down.

b) Calculate the time taken to hit the ground.
Solution: $t = ?$

$$v_f = v_i + gt$$

$$t = (v_f - v_i)/g = (-17.3 - 10)/(-9.8)\ \text{s} \approx 2.73\ \text{s}$$

2.5 MOTION GRAPHS

Obtaining average and instantaneous velocity from a graph of position versus time: Average velocity between two points of a position versus time graph can be obtained as the slope of the line joining the two points. For instantaneous velocity the two points need to be infinitesimally close to each other and the line joining the two points becomes the tangent line at the given point. Therefore instantaneous velocity at a given point of a position versus time graph can be obtained as the slope of the line tangent to the curve at the given point.

Example: The following is a graph of position versus time for a certain particle.

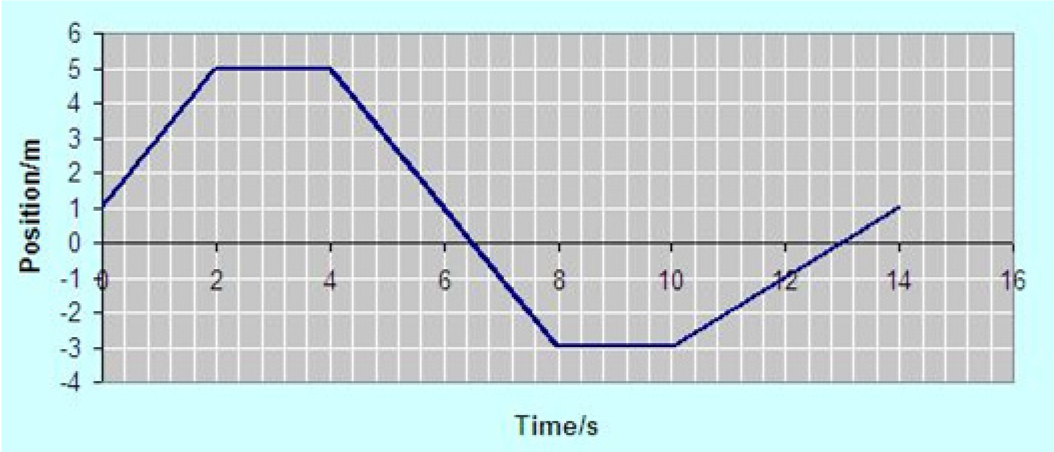

Figure 2.1

a) What is the initial position of the particle?
Solution: Initial position is position at $t = 0$.

$t = 0$; $x = ?$

$x = 1$ m

b) Calculate the average velocity between $t = 2$ s and $t = 10$ s.
Solution: The average velocity is equal to the slope of the line joining the points (2 s, 5 m) and (10 s, -3 m).

$(t_i, x_i) = (2$ s, 5 m$)$; $(t_f, x_f) = (10$ s, -3 m$)$; $v_{av} = ?$

$v_{av} = (x_f - x_i)/(t_f - t_i) = (-3 - 5)/(10 - 2)$ m/s $= -1$ m/s

c) Calculate the average velocity for the entire trip.
The average velocity for the entire trip is the slope of the line joining the points (0, 1 m) and (14 s, 1 m).

$(t_i, x_i) = (0, 1$ m$)$; $(t_f, x_f) = (14$ s, 1 m$)$; $v_{av} = ?$

$v_{av} = (x_f - x_i)/(t_f - t_i) = (1 - 1)/(14 - 0)$ m/s $= 0$

d) Calculate the displacement of the particle between $t = 3$ s and $t = 9$ s.
Solution: $x_i = 5$ m; $x_f = -3$ m; $\Delta x = ?$

$\Delta x = x_f - x_i = (-3 - 5)$ m $= -8$ m

e) On what interval(s) of time is the particle

i. At rest.

Solution: When the particle is at rest, the graph of position versus time should be horizontal. Therefore the particle is at rest on the time intervals between $t = 2$ s and $t = 4$ s and between $t = 8$ s and $t = 10$ s.

ii. Moving to the right?

Solution: When moving to the right its velocity (slope) should be positive. Therefore the particle is moving to the right on the time intervals between $t = 0$ and $t = 2$ s and between $t = 10$ s and $t = 14$ s.

iii. Moving to the left?

Solution: When moving to the left its velocity (slope) should be negative. Therefore the particle is moving to the left on the time interval between $t = 4$ s and $t = 8$ s

f) What is the instantaneous velocity of the particle at $t = 1$ s
Solution: If the graph of position versus time is a straight line, average and instantaneous velocity are the same. Since the graph is a straight line between $t = 0$ and $t = 2$ s, the instantaneous velocity at $t = 1$ s is equal to the average velocity between $t = 0$ and $t = 2$ s.

Solution: $(t_i, x_i) = (0, 1$ m$)$; $(t_f, x_f) = (2$ s , 5 m$)$; $v = v_{av} = ?$

$$v = (x_f - x_i)/(t_f - t_i) = (5 - 1)/(2 - 0) \text{ m/s} = 2 \text{ m/s}$$

Obtaining average and instantaneous acceleration and displacement from a graph of velocity versus time: Average acceleration between two points of a velocity versus time graph can be obtained as the slope of the line joining the two points. Instantaneous acceleration at a given point of a velocity versus time graph can be obtained as the slope of the line tangent to the graph at the given point. Displacement can be obtained from a graph velocity versus time as the area enclosed between the velocity versus time curve and the time axis. Areas above the time axis are taken to be positive while areas below the time axis are taken to be negative.

Example: The following is a graph of velocity versus time for a certain particle.

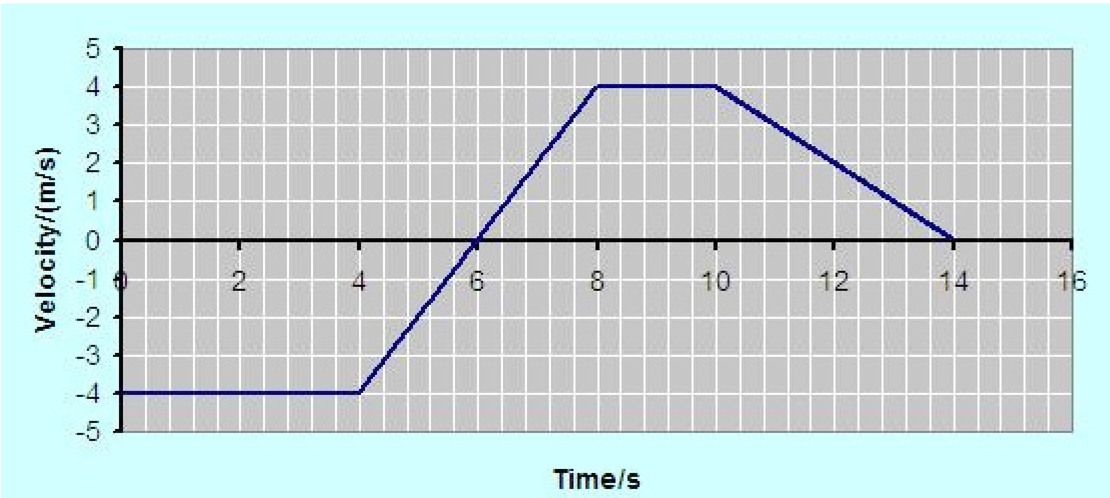

Figure 2.2

a) Calculate the average acceleration for the first ten seconds.
Solution: The average acceleration in the first ten seconds is equal to the slope of the line joining the events $(0, -4$ m/s$)$ and $(10$ s, 4 m/s$)$.

Solution: $(t_i, v_f) = (0, -4$ m/s$)$; $(t_f, v_f) = (10$ s, 4 m/s$)$; $a_{av} = ?$

$$a_{av} = (v_f - v_i)/(t_f - t_i) = (4 - (-4))/(10 - 0) \text{ m/s}^2 = 0.8 \text{ m/s}^2$$

b) Calculate the average acceleration for the entire trip.
Solution: The average acceleration for the entire trip is equal to the slope of the line joining the points (*0, -4* m) and (*14* s, *0*).

Solution: $(t_i\ ,\ v_f) = (0,\ -4\ \text{m/s})\ ;\ (t_f\ ,v_f) = (14\ \text{s},\ 0)\ ;\ a_{av} = ?$

$$a_{av} = (v_f - v_i)/(t_f - t_i) = (0 - (-4))/(14 - 0)\ \text{m/s}^2 = 0.29\ \text{m/s}^2$$

c) On what interval(s) of time is the particle
  i. Moving to the right?
  Solution: The particle is moving to the right when the velocity is positive. Therefore the particle is moving to the right on the interval between $t = 6$ s and $t = 14$ s.

  ii. Moving to the left?
  Solution: The particle is moving to the left when the velocity is negative. Therefore the particle is moving to the left on the interval between $t = 0$ and $t = 6$ s.

  iii. Temporarily at rest?
  Solution: The particle is temporarily at rest when the velocity is zero. Therefore the particle is temporarily at rest at $t = 6$ s and $t = 14$ s.

d) On what interval(s) of time is the particle
  i. Moving with a constant velocity?
  Solution: The particle moves with a constant velocity when its acceleration is zero or when the graph of velocity versus time is a horizontal line. Therefore the particle is moving with a constant velocity on the time intervals between $t = 4$ s and $t = 8$ s and between $t = 8$ s and $t = 10$ s.

  ii. Increasing its velocity?
  Solution: The velocity of the particle increases when the acceleration or slope is positive. Therefore its velocity is increasing on the interval between $t = 4$ s and $t = 8$ s.

  iii. Decelerating?
  Solution: The particle decelerates (its velocity decreases) when its acceleration or slope is negative. Therefore it is decelerating on the time interval between $t = 10$ s and $t = 14$ s.

e) Calculate its instantaneous acceleration at $t = 2$ s.
Solution: At $t = 2$ s, the tangent line is horizontal and the slope of a horizontal line is zero. Therefore the instantaneous acceleration at $t = 2$ s is zero.

Practice Quiz 2.2

Choose the best answer

1. A particle, starting from a speed of 28 m / s was accelerated for 18 seconds with an acceleration of 8 m / s ^{2}. How far did it travel?
   A) 1324 m
   B) 432 m
   C) 3096 m
   D) 1800 m
   E) 360 m

2. A particle was uniformly accelerated from a speed of 22.5 m / s to a speed of 43.8 m / s in 5 seconds. How far did it travel?
   A) 46.647 m
   B) 165.75 m
   C) 331.5 m
   D) 53.25 m
   E) 145.197 m

3. The speed of a uniformly accelerated particle changed from *1200* m / s to *500* m/s in a distance of *2050* m. Its acceleration and the time taken respectively are
   A) *-0.171* m / s ^{2} and *1.206* s
   B) *-0.171* m / s ^{2} and *2.412* s
   C) *-290.244* m / s ^{2} and *2.412* s
   D) *-290.244* m / s ^{2} and *1.206* s
   E) *412.195* m / s ^{2} and *1.206* s

4. An object was released from a height of *80* m. Its velocity just before it hits the ground and the time taken to hit the ground respectively are
   A) *-1568* m / s and *16.327* s
   B) *-28* m / s and *2.857* s
   C) *-39.598* m / s and *4.041* s
   D) *-28* m / s and *4.041* s
   E) *-39.598* m / s and *2.857* s

5. An object thrown upwards reaches a maximum height of *5* m. The speed with which it was thrown and the time take to reach the maximum height respectively are
   A) *9.899* m / s and *1.01* s
   B) *7* m / s and *1.429* s
   C) *98* m / s and *39.2* s
   D) *9.899* m / s and *1.429* s
   E) *7* m / s and *1.01* s

6. An object is thrown upwards from a *32* m building with a speed of *39* m / s. Its velocity just before it hits the ground and the time taken to hit the ground respectively are
   A) *-46.349* m / s and *8.709* s
   B) *-46.349* m / s and *6.613* s
   C) *-25.811* m / s and *8.709* s
   D) *-42.832* m / s and *8.159* s
   E) *-25.811* m / s and *6.613* s

7. Consider Figure 2.1. Calculate the average velocity between the third and the ninth seconds.
   A) *-1.333* m / s
   B) *1.333* m / s
   C) *-0.333* m / s
   D) *-0.75* m / s
   E) *0.75* m / s

8. Consider Figure 2.1. On what interval(s) of time is it moving to the left?
   A) *0-8* s
   B) *4-8* s
   C) *0-2* s, and *10-14* s
   D) *0-2* s, *4-8* s, and *10-14* s
   E) *10-14* s

9. Consider Figure 2.1. Calculate the instantaneous velocity after *1* second.
   A) *0.5* m / s
   B) *1* m / s
   C) *0* m / s
   D) *2* m / s
   E) *-0.5* s

10. Consider Figure 2.2. Calculate its average acceleration for the first 8 seconds.
    A) *-1.5* m / s ^{2}
    B) *0.5* m / s ^{2}
    C) *1* m / s ^{2}
    D) *2* m / s ^{2}
    E) *-0.5* m / s ^{2}

11. Consider Figure 2.2. Calculate its displacement between the *12* th and *14* th seconds.
    A) *6* m
    B) *3* m
    C) *1* m
    D) *2* m
    E) *4* m

3 VECTORS

Your goal for this chapter is to learn about the mathematics needed to describe physical quantities that have direction.

Physical quantities are classified into two based on how they are represented. Physical quantities that can be represented completely by a number and a unit are called *scalars*. Examples are mass (2 kg), time (6 hour), length (7 m), and temperature (20°C). On the other hand, physical quantities that require specification of direction in addition to a number and unit are called *vectors*. Examples are displacement (2 m east), velocity (5 m/s west), acceleration, force (6 N south) and area. Symbolically, vectors are represented by a capital letter with an arrow on top.

3.1 ADDING VECTORS GRAPHICALLY

Graphically, a vector is represented by means of an arrow. The direction of the vector is represented by means of the direction of the arrow. The magnitude (numerical value) of the vector is represented by the length of the vector. The length of the arrow is drawn in such a way that it is proportional to the magnitude of the vector. To this end, a scale relating the magnitude of the vector and the length of the vector needs to be specified. The direction of a vector is stated by specifying the angle measured with respect to a certain reference line. For example, the direction of the vector '2 m 30° west of north' is specified with respect to north. The *default reference line* is the positive x-axis or east (i.e. horizontal line to the right of the tail of the vector). That means if no reference line is specified, the reference line is assumed to be the positive x-axis. Angles measured in a counter clockwise direction from this reference line are taken to be positive, while angles measured in a clockwise direction are taken to be negative.

Multiplying a vector by a constant has the effect of multiplying the magnitude of the vector by the constant. If the constant is positive, the direction remains the same, and if the constant is negative, the direction becomes opposite.

The sum of two or more vectors is the single vector with the same effect. To add vectors graphically, first join the vectors head to tail. Then the sum vector is the vector whose tail is the tail of the first vector and whose head is the head of the last vector. To subtract vector $\vec{B}$ from vector $\vec{A}$, add the negative of $\vec{B}$ to $\vec{A}$ and then use rules of addition. The negative of a vector is the vector with the same magnitude but opposite direction.

3.2 ADDING VECTORS ANALYTICALLY

Components of a Vector: Any vector can be written as a sum of a horizontal vector and a vertical vector. The horizontal vector is called horizontal or x-component $\left(\vec{A}_x\right)$ of the vector and the vertical vector is called the vertical or y-component $\left(\vec{A}_y\right)$ of the vector. If the magnitude of vector $\vec{A}$ is A and its default angle measure (angle measured with respect to the positive x-axix) is θ, then using the definitions of a cosine and sine based on a right angled traingle, the components can be expressed in terms of magnitude and direction as

$$A_x = A\cos(\theta)$$
$$A_y = A\sin(\theta)$$

The components A_x and A_y can be positive or negative. The x-component (A_x) is positive if $\vec{A}_x$ is to the right and negative if $\vec{A}_x$ is to the left. The y-component (A_y) is positive if $\vec{A}_y$ is up and negative if $\vec{A}_y$ is down. If the default angle is used, the cosine and sine will result in the correct sign for the x and y component, respectively.

Example: In each of the following calculate the x and y components of the vector.

a) $\vec{A} = 100$ m 53° north of east.

Solution:

$A = 100$ m; $\theta = 53°$; $A_x = ?$; $A_y = ?$

$$A_x = A\cos(\theta) = 100\cos(53°)\ \text{m} = 60\ \text{m}$$

$$A_y = A\cos(\theta) = 100\sin(53°)\ \text{m} = 80\ \text{m}$$

b) $\vec{A} = 10$ m 30° south of east.

Solution:

$A = 10$ m; $\theta = 180° + 30° = 210°$; $A_x = ?$; $A_y = ?$

$$A_x = A\cos(\theta) = 10\cos(210)\ \text{m} = -8.7\ \text{m}$$

$$A_y = A\cos(\theta) = 10\sin(210)\ \text{m} = -5\ \text{m}$$

Obtaining magnitude and direction of a vector from its components: The magnitude of a vector can be obtained from its components using pythagorean theoerm.

$$A = \sqrt{A_x^2 + A_y^2}$$

The direction of a vector can be obtained from its components by using the definition of a tangent for a right angled triangle; That is, $\tan(\theta) = \frac{A_x}{A_y}$ or

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

To obtain the default angle (angle measured with respect to the positive x-axis), this formula has to be modified. The periodicity of tangent is 180° and not 360°. The calculator will give you angles in the range between -90° and 90° only (that is, angles in the $x > 0$ quadrants). For angles that fall in the range between 90° and 270°(that is, angles in the $x < 0$ quadrants) the formula has to be modified by adding 180°. For example the calculator will give you the same angle for a vector whose x and y components are 3 and 4 and a vector whose components are

-3 and -4: $\theta = \tan^{-1}\left(\frac{3}{4}\right) = \tan^{-1}\left(\frac{-3}{-4}\right) = 37°$. But obviously, the default angle for the later is 37° + 180° = 217°. Further, the calculator will give you an error when $A_x = 0$. Therefore, for the default angle, the formulas has to be modified as follows.

$$\theta = \begin{cases} \tan^{-1}\left(\frac{A_y}{A_x}\right) \text{ if } A_x > 0 \\ \tan^{-1}\left(\frac{A_y}{A_x}\right) + 180° \text{ if } A_x < 0 \\ 90° \text{ if } A_x = 0 \text{ and } A_y > 0 \\ -90° \text{ if } A_x = 0 \text{ and } A_y < 0 \end{cases}$$

Example: Find the magnitude and direction of a vector whose compnents are

a) 3 m and 4 m

Solution:

$A_x = 3\text{ m}; A_y = 4\text{ m}; A = ?;\ \theta = ?$

$$A = \sqrt{A_x^{\,2} + A_y^{\,2}} = \sqrt{3^2 + 4^2}\ \text{m} = 5\text{ m}$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) = \tan^{-1}\left(\frac{4}{3}\right) = 53°$$

b) -3 m and 4 m

Solution:

$A_x = -3\text{ m}; A_y = 4\text{ m}; A = ?;\ \theta = ?$

$$A = \sqrt{A_x^{\,2} + A_y^{\,2}} = \sqrt{(-3)^2 + 4^2}\ \text{m} = 5\text{ m}$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) = \tan^{-1}\left(\frac{4}{-3}\right) = -53° + 180^{\circ} = 127°$$

Adding Vectors Analytically: Using graphical addition of vectors, it can be shown that the x and y components of the sum of two or more vectors are equal to the sum of the x-components and y-components of the vectors being added respectively; That is, if $\vec{R} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \dots,$ then

$$R_x = A_{1x} + A_{2x} + A_{3x} + \dots = \sum A_{ix}$$
$$R_y = A_{1y} + A_{2y} + A_{3y} + \dots = \sum A_{iy}$$

Once the components of the sum vector are obtained, then the magnitude and the default angle of the sum vector can be calculated using the above formulas.

Example: Given the vector $\vec{A} = 10\text{ m}\,45°\,\text{west of North}$ and $\vec{B} = 20\text{ m}\ 37°\text{east of North}$, Find the magnitude and direction of the sum vector $\vec{A} + \vec{B}$.

Solution:

$A = 10\text{ m};\ \theta_A = 90° + 45° = 135°;\ B = 20\text{ m};\ \theta_B = 37°;\ R = ?;\ \theta = ?$

$$A_x = A\cos(\theta_A) = 10\cos(135°)\ \text{m} = -7.07\text{ m}$$

$$A_y = A\sin(\theta_A) = 10\sin(135°)\ \text{m} = 7.07\text{ m}$$

$$B_x = B\cos(\theta_B) = 20\cos(37)\ \text{m} = 16\text{ m}$$

$$B_y = B\cos(\theta_B) = 20\sin(37)\ \text{m} = 12\ \text{m}$$

$$R = \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2} = \sqrt{(-7.07+16)^2 + (7.07+12)^2}\ \text{m} = 21.06\ \text{m}$$

$$\theta = \tan^{-1}\left(\frac{A_y + B_y}{A_x + B_x}\right) = \tan^{-1}\left(\frac{7.07+12}{-7.07+16}\right) = 64.9°$$

Practice Quiz 3.1

Choose the best answer

1. Which of the following physical quantities is a scalar?
   A) Velocity
   B) Force
   C) Length
   D) Displacement
   E) Acceleration

2. The default angle (angle measured with respect to the positive x-axis) for the vector ***A*** = *2* m north is
   A) *90* deg
   B) *-90* deg
   C) *0* deg
   D) *270* deg
   E) *180* deg

3. The default angle (angle measured with respect to the positive x-axis) for the vector ***A*** = *8* m *10* deg south of east is
   A) *10* deg
   B) *100* deg
   C) *80* deg
   D) *-10* deg
   E) *170* deg

4. Using the scale *1* cm for *10* m/s, the vector ***A*** = *48* m/s east, can be graphically represented by an arrow of length
   A) *0.208* cm
   B) *48* cm
   C) *9.6* cm
   D) *4.8* cm
   E) *2.4* cm

5. If the vector ***A*** = *100* m/s *15* deg north of east, using the scale *1* cm for *40* m/s, the vector *2**A*** can be graphically represented by an arrow of length _________ and default angle (angle measured with respect to the positive x-axis) ___________.
   A) *5* cm, *15* deg
   B) *10* cm, *15* deg
   C) *0.2* cm, *75* deg
   D) *10* cm, *-15* deg
   E) *5* cm, *-15* deg

6. If the vector ***A*** = *60* m/s east, using the scale *1* cm for *20* m/s, the vector *-0.5**A*** can be graphically represented by an arrow of length _________ and default angle (angle measured with respect to the positive x-axis) ___________.
   A) *1.5* cm, *180* deg
   B) *3* cm, *0* deg
   C) *1.5* cm, *0* deg
   D) *0.333* cm, *90* deg
   E) *3* cm, *180* deg

7. Given the vectors ***A*** = *4* m east and ***B*** = *7* m north, using graphical addition determine which of the following most likely represent the magnitude and direction of the sum vector (***A*** + ***B***) respectively. (You should be able to answer it with sketches not drawn to scale.)
   A) *8.062* m, *240.255* deg
   B) *8.062* m, *119.745* deg
   C) *11* m, *60.255* deg
   D) *11* m, *119.745* deg
   E) *8.062* m, *60.255* deg

8. Given the vectors ***A*** = *4* m west and ***B*** = *11* m south, using graphical method determine which of the following most likely represent the magnitude and direction of the difference vector ***A*** - ***B*** respectively. (You should be able to answer it with sketches not drawn to scale.)
   A) *15* m, *109.983* deg
   B) *11.705* m, *70.017* deg
   C) *15* m, *70.017* deg
   D) *11.705* m, *109.983* deg
   E) *11.705* m, *250.017* deg

9. Determine the x and y components of the vector ***A*** = *10* m west.
   A) (*-10* m, *10* m)
   B) (*0* m, *10* m)
   C) (*10* m, *0* m)
   D) (*0* m, *-10* m)
   E) (*-10* m, *0* m)

10. Determine the x and y components of the vector ***A*** = *110* m *80* deg south of west.
   A) (*-12.023* m, *-138.6* m)
   B) (*-14.865* m, *-23.474* m)
   C) (*-19.101* m, *-108.329* m)
   D) (*-14.865* m, *-108.329* m)
   E) (*-19.101* m, *-23.474* m)

11. Calculate the magnitude of a vector whose x and y components are *2* m and *5* m respectively.
    A) *29* m
    B) *5.385* m
    C) *3* m
    D) *2.646* m
    E) *7* m

12. Calculate the direction (angle with respect to the positive x axis) of a vector whose x and y components are *16* m and *15* m respectively.
    A) *46.848* deg
    B) *65.53* deg
    C) *84.674* deg
    D) *9.919* deg
    E) *43.152* deg

13. Calculate the magnitude and direction (with respect to the positive x axis) of a vector whose x and y components are *-4* m and *21* m respectively.
    A) *21.378* m, *95.784* deg
    B) *23.515* m, 100.784 deg
    C) *21.378* m, *100.784* deg
    D) *21.378* m, *-79.216* deg
    E) *23.515* m, *95.784* deg

14. Given the vectors ***A*** = *30* m *50* deg north of east and ***B*** = *60* m *40* deg north of west, find the x and y components of their sum vector ***A*** + ***B***.
    A) (*-17.721* m, *71.767* m)
    B) (*-13.037* m, *61.549* m)
    C) (*-13.037* m, *11.148* m)
    D) (*-26.679* m, *61.549* m)
    E) (*-26.679* m, *11.148* m)

15. Given the vectors ***A*** = *50* m *70* deg north of west and ***B*** = *20* m *20* deg north of west, find the magnitude and direction (with respect to the positive x axis) of their sum vector ***A*** + ***B***.
    A) *117.883* m, *123.699* deg
    B) *64.696* m, *123.699* deg
    C) *64.696* m, *165.53* deg
    D) *64.696* m, *201.409* deg
    E) *117.883* m, *165.53* deg

3.3 UNIT VECTORS

Vector operations can be simplified greatly if the vectors are represented in terms of orthogonal unit vectors. A *unit vector* is a vector whose magnitude is one. A unit vector in the direction of vector $\vec{A}$ $(\vec{e}_A)$ is obtained by dividing the vector by its magnitude (A).

$$\vec{e}_A = \frac{\vec{A}}{A}$$

Unit vectors in the direction of the positive x-axis and y-axis are customarily represented by $\hat{i}$ and $\hat{j}$ respectively. Therefore the horizontal and vertical components of a vector can be written in terms of these unit vectors as $\vec{A}_x = A_x\hat{i}$ and $\vec{A}_y = A_y\hat{j}$ respectively. Any vector can be written as the sum of its x and y components. Therefore the vector can be represented in terms of the unit vectors $\hat{i}$ and $\hat{j}$ as

$$\vec{A} = A_x\hat{i} + A_y\hat{j} = A\cos(\theta)\hat{i} + A\sin(\theta)\hat{j}$$

Example: Express the following vectors in terms of the $\hat{i}$ and $\hat{j}$ unit vectors

a) $\vec{A} = 10\text{ m } 37°\text{ south of west}$
Solution:
$A = 10\text{ m};\ \theta = 180° + 37° = 217°$

$$\vec{A} = A\cos\theta\hat{i} + A\sin\theta\hat{j} = 10\text{ m }\cos(217°)\hat{i} + 10\text{ m }\sin(217°) = (-8\hat{i} - 6\hat{j})\text{ m}$$

b) $\vec{A} = 2\text{ m west}$
Solution:
$A = 2\text{ m};\ \theta = 180°$

$$\vec{A} = 2\text{ m}\cos(180°)\hat{i} + 2\text{ m}\sin(180°) = -2\hat{i}\text{ m}$$

Since the coefficient of is the x-component and the coefficient of is the y-component $\left(\text{That is } \vec{A} = A_x\hat{i} + A_y\hat{j}\right)$, the magnitude and direction of a vector expressed in terms of $\hat{i}$ and $\hat{j}$ unit vectors can be obtained using Pythagorean theorem and the definition of tangent as shown earlier.

Example: Find the magnitude and direction of the following vectors expressed in terms of $\hat{i}$ and $\hat{j}$ unit vectors.

a) $\vec{A} = -3\hat{i} + 4\hat{j}$
Solution:
$A_x = -3;\ A_y = 4;\ A = ?;\ \theta = ?$

$$A = \sqrt{A_x^{\ 2} + A_y^{\ 2}} = \sqrt{(-3)^2 + 4^2} = 5$$

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right) + 180° = \tan^{-1}\left(\frac{4}{-3}\right) + 180° = -53° + 180° = 127°$$

b) $\vec{A} = 5\hat{i}$
Solution:
$A_x = 5;\ A_y = 0;\ A = ?;\ \theta = ?$

$$A = \sqrt{A_x^{\ 2} + A_y^{\ 2}} = \sqrt{5^2 + 0^2} = 5$$

$$\theta = \tan^{-1}\left(\frac{0}{5}\right) = 0°$$

Operating with vectors Expressed in terms of $\hat{i}$ and $\hat{j}$ *unit vectors*: The general rule is to treat vectors in the $\hat{i}-\hat{j}$ notation like any algebraic expression while adding (subtracting) or multiplying by a constant provided the $\hat{i}$ and $\hat{j}$ unit vectors are treated as independent algebraic entities ($\hat{i}$ terms can be combined with $\hat{i}$ terms and $\hat{j}$ terms can be combined with $\hat{j}$ terms only); That is,

$$C\left[A_x\hat{i}+A_y\hat{j}\right]=CA_x\hat{i}+CA_y\hat{j}$$

$$\left[A_x\hat{i}+A_y\hat{j}\right]+\left[B_x\hat{i}+B_y\hat{j}\right]=\left(A_x+B_x\right)\hat{i}+\left(A_y+B_y\right)\hat{j}$$

Where C is a constant.

Example: Find the magnitude and direction of the following if $\vec{A}=-2\hat{i}+3\hat{j}$, $\vec{B}=4\hat{j}$ and $\vec{C}=4\hat{i}-2\hat{j}$

a) $2\vec{A}+\vec{B}$
Solution:

$$2\vec{A}+\vec{B}=2\left(-2\hat{i}+3\hat{j}\right)+\left(4\hat{j}\right)=-4\hat{i}+6\hat{j}+4\hat{j}=-4\hat{i}+10\hat{j}$$

$$A=\sqrt{A_x^{\,2}+A_y^{\,2}}=\sqrt{(-4)^2+10^2}=10.8$$

$$\theta=\tan^{-1}\left(\frac{A_y}{A_x}\right)+180°=\tan^{-1}\left(\frac{10}{-4}\right)+180°=111.8°$$

b) $3\vec{A}-4\vec{B}-2\vec{C}$
Solution:

$$3\vec{A}-4\vec{B}-2\vec{C}=3\left(-2\hat{i}+3\hat{j}\right)-4\left(4\hat{j}\right)-2\left(4\hat{i}-2\hat{j}\right)=-14\hat{i}-3\hat{j}$$

$$\left|3\vec{A}-4\vec{B}-2\vec{C}\right|=\sqrt{(-14)^2+(-3)^2}=14.3$$

$$\theta=\tan^{-1}\left(\frac{-3}{-14}\right)+180°=192.1°$$

A three dimensional vector can written as a sum of vectors along the x-axis, y-axis, and z-axis $\vec{A}=\vec{A}_x+\vec{A}_y+\vec{A}_z$. The unit vector along the z-axis is customarily represented by $\hat{k}$. Thus the vector $\vec{A}$ can be written in the $\hat{i}-\hat{j}-\hat{k}$ notation as

$$\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$$

Where A_x, A_y and A_z are components of $\vec{A}$ along the x, y, and z axis respectively? If A_{xy} is the projection of a 3-dimensional vector on the xy-plane, then $A_{xy} = \sqrt{A_x{}^2 + A_y{}^2}$. And since the projection on the xy-plane and the z-component are perpendicular to each other, the magnitude of the vector is equal to $\sqrt{A_{xy}{}^2 + A_z{}^2}$. Therefore the magnitude of a 3-dimensional vector is given as

$$A = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}$$

3.4 DOT PRODUCT

The dot product of two vectors $\vec{A}$ and $\vec{B}$ $\left(\vec{A} \cdot \vec{B}\right)$ is defined to be the product of the magnitude of vector $\vec{A}$ and the component of vector $\left(B_{\mathrm{P}}\right)$ in the direction of vector $\vec{A}$; That is, $\vec{A} \cdot \vec{B} = AB_{\mathrm{P}}$. And if the angle between $\vec{A}$ and $\vec{B}$ is , then $B_{\mathrm{P}} = B\cos\theta$ and

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

The following are some properties of a dot product.

1) The dot product of a vector with itself is equal to the square of its magnitude.

$$\vec{A}\cdot\vec{A}=A^2$$

2) The dot product of two vectors $\vec{A}$ and $\vec{B}$ that are perpendicular to each other is zero.

$$\vec{A}\cdot B=0 \text{ if } \vec{A}\perp\vec{B}$$

3) Dot product is commutative

$$\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$$

4) Dot product is distributive over addition.

$$\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$$

Example: Calculate $\vec{A}\cdot\vec{B}$ if $\vec{A}=8$ m east and $\vec{B}=4$ m 37° west of north

Solution:

$A=8$ m; $B=4$ m; $\theta=90°+37°=127°$; $\vec{A}\cdot\vec{B}=?$

$$\vec{A}\cdot\vec{B}=AB\cos\theta=(8\text{ m})(4\text{ m})\cos(127°)=-7.2\text{ m}^2$$

Dot Product in terms of x-y-z components: Since the magnitude of a unit vector is one a unit vector dotted to itself is equal to one. The dot product between two different Cartesian unit vectors is zero because they are perpendicular to each other.

$$\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\vec{k}\cdot\hat{k}=1$$
$$\hat{i}\cdot\hat{j}=\hat{\hat{i}}\cdot\hat{k}=\hat{j}\cdot\hat{k}=0$$

Now if $\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ and $\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$, then $\vec{A}\cdot\vec{B}=\left(\vec{A}_x\hat{i}+\vec{A}_y\hat{j}+\vec{A}_z\hat{k}\right)\cdot\left(\vec{B}_x\hat{i}+\vec{B}_y\hat{j}+\vec{B}_z\hat{k}\right)$ and by direct expansion it can be shown that

$$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z$$

The angle between two vectors may be obtained by equating the two different forms of a dot product; That is $\vec{A}\cdot\vec{B} = AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$ or $\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{AB}$. Solving for θ, the angle between the two vectors can be calculated from

$$\theta = \cos^{-1}\left(\frac{A_xB_x + A_yB_y + A_zB_z}{AB}\right)$$

Example: Given $\vec{A} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{B} = -4\hat{i} + \hat{j} - \hat{k}$, Calculate $\vec{A}\cdot\vec{B}$

Solution:

$A_x = 2;\ A_y = -3;\ A_z = 4;\ B_x = -4;\ B_y = 1;\ B_z = -1$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z = 2(-4) + (-3)(1) + (4)(-1) = -15$$

Example: Given the vectors $\vec{A} = 2\hat{i} - 4\hat{j}$ and $\vec{B} = \hat{i} + 3\hat{j}$, Calculate the angle formed between $\vec{A}$ and $\vec{B}$.

Solution:

$A_x = 2;\ A_y = -4;\ A_z = 0;\ B_x = 1;\ B_y = 3;\ B_z = 0;\ \theta = ?$

$$A = \sqrt{A_x^2 + A_y^2} = \sqrt{2^2 + (-4)^2} = \sqrt{20}$$

$$B = \sqrt{B_x^2 + B_y^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\theta = \cos^{-1}\left(\frac{A_xB_x + A_yB_y + A_zB_z}{AB}\right) = \cos^{-1}\left(\frac{(2)(1) + (-4)(1)}{\sqrt{20}\sqrt{10}}\right) = 135°$$

3.5 CROSS PRODUCT

The cross product between two vectors $\vec{A}$ and $\vec{B}$, written as $\vec{A}\times\vec{B}$, is a vector whose magnitude is equal to the area of the parallelogram determined by the two vectors and whose direction is perpendicular to the plane determined by the two vectors. If the angle between the vectors is θ, then its magnitude $\left(\left|\vec{A}\times\vec{B}\right|\right)$ is given as

$$\left|\vec{A}\times\vec{B}\right| = AB\sin\theta$$

Where A and B are the magnitudes of vectors $\vec{A}$ and $\vec{B}$ respectively. The direction can be either perpendicularly out of the plane, represented by a dot $(\cdot)$, or perpendicularly into the plane, represented by a cross $(\times)$. To distinguish between these two possibilities either the screw rule or the right hand rule can be used.

The Screw Rule: First join the vectors tail to tail. Then place a screw at the intersection of the two vectors perpendicularly to the plane determined by the two vectors. To find the direction of $\vec{A}\times\vec{B}$ rotate the screw from towards (and to find direction of $\vec{B}\times\vec{A}$ rotate from $\vec{B}$ towards $\vec{A}$). Then the direction of movement of the screw gives the direction of $\vec{A}\times\vec{B}$.

The Right Hand Rule: First align the thumb in such a way that it is perpendicular to the plane determined by the index finger and middle finger of the right hand. To find the direction of $\vec{A}\times\vec{B}$, If $\vec{A}$ is represented by the index finger and $\vec{B}$ is represented by the middle finger, then thumb represents the direction of $\vec{A}\times\vec{B}$.

The following are some properties of a cross product.

1) The cross product of a vector with itself is zero.

$$\vec{A}\times\vec{A}=0$$

2) Cross product is not commutative.

$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$$

3) Cross product is distributive over addition.

$$\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{A}\times\vec{C}$$

Example: If $\vec{A}=2$ m east and $\vec{B}=4$ m north and

a) Find the magnitude and direction of $\vec{A}\times\vec{B}$
Solution:
$A=2$ m; $B=4$ m; $\theta=90°$

$$\left|\vec{A}\times\vec{B}\right|=AB\sin\theta=2\times4\sin\left(90°\right)\ \text{m}^2=8\ \text{m}^2$$

As can be shown from the screw rule or the right hand rule, the direction is perpendicularly out $(\cdot)$.

b) Find the magnitude and direction of $\vec{B}\times\vec{A}$.
Solution: $\vec{B}\times\vec{A}$ has same magnitude but opposite direction to that of $\vec{A}\times\vec{B}$.

$$\left|\vec{B}\times\vec{A}\right|=8\ \text{m}^2$$

Its direction perpendicularly in $(\times)$.

The cross product in terms of the $\hat{i}-\hat{j}-\hat{k}$ Cartesian unit vectors: As can be shown using the basic definitions of a cross product

$$\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$$
$$\hat{i}\times\hat{j}=\hat{k};\hat{j}\times\hat{k}=\hat{i};\ \hat{k}\times\hat{i}=\hat{j}$$
$$\hat{j}\times\hat{i}=-\hat{k};\ \hat{k}\times\hat{j}=-\hat{i};\ \hat{i}\times\hat{k}=-\hat{j}$$

Now, if $\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ and $\vec{B}=B_x\hat{i}+B_y\hat{j}+B_z\hat{k}$, then $\vec{A}\times\vec{B}=\left[A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right]\times\left[B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right]$. Direct expansion of this cross product results in the following expression for the cross product.

$$\vec{A}\times\vec{B}=\hat{i}\left(A_yB_z-A_zB_y\right)+\hat{j}\left(A_zB_x-A_xB_z\right)+\hat{k}\left(A_xB_y-A_yB_x\right)$$

An expression for the angle between two vectors can be obtained by equating the two different expressions for $\left|\vec{A}\times\vec{B}\right|$; That is, $\left|\vec{A}\times\vec{B}\right| = AB\sin(\theta) = \sqrt{\left(A_yB_z - A_zB_y\right)^2 + \left(A_zB_x - A_xB_z\right)^2 + k\left(A_xB_y - A_yB_x\right)^2}$ and solving for θ, the following expression for the angle between the vectorsz

$$\theta = \sin^{-1}\left(\frac{\sqrt{\left(A_yB_z - A_zB_y\right)^2 + \left(A_zB_x - A_xB_z\right)^2 + k\left(A_xB_y - A_yB_x\right)^2}}{AB}\right)$$

Practice Quiz 3.2

Choose the best answer

1. Which of the following statements is incorrect?
   A) The x-component of the unit vector ***k*** is zero.
   B) The y-component of the unit vector ***j*** is one.
   C) The unit vectors ***i*** and ***j*** are perpendicular to each other.
   D) The magnitude of any unit vector is one.
   E) The x-component of the unit vector ***i*** is zero.

2. Which of the following statements is correct?
   A) The dot product between any two parallel vectors is zero.
   B) ***A*** × ***B*** and ***B*** × ***A*** are equal
   C) The cross product between any two perpendicular vectors is zero.
   D) The cross product between two vectors is a scalar.
   E) The dot product between two vectors is a scalar quantity.

3. Express the vector ***A*** = *4.3* m *67°* south of east in the ***i-j*** notation.
   A) *2.45* m ***i*** + *-3.958* m ***j***
   B) *1.68* m ***i*** + *-1.232* m ***j***
   C) *1.68* m ***i*** + *-3.958* m ***j***
   D) *1.68* m ***i*** + *-7.408* m ***j***
   E) *2.128* m ***i*** + *-3.958* m ***j***

4. Calculate the magnitude and the direction of the vector ***A*** = *-4.1* m ***i*** + *4.8* m ***j*** respectively.
   A) *6.313* m, *130.503°*
   B) *2.403* m, *212.092°*
   C) *2.403* m, *130.503°*
   D) *7.641* m, *108.246°*
   E) *6.313* m, *212.092°*

5. Given the vectors

   $\boldsymbol{A} = 1.5\,\boldsymbol{i} - 4.7\,\boldsymbol{j} - 3.1\,\boldsymbol{k}$

   $\boldsymbol{B} = 6.2\,\boldsymbol{i} - 1.5\,\boldsymbol{j} + 6.2\,\boldsymbol{k}$

   Calculate the magnitude of $\boldsymbol{A} - \boldsymbol{B}$.

   A) *14.535*
   B) *10.9*
   C) *17.765*
   D) *9.084*
   E) *18.855*

6. Given the vectors

   $\boldsymbol{A} = 7.4$ N $45°$ south of east and

   $\boldsymbol{B} = 3.1$ m $20°$ east of south,

   Calculate $\boldsymbol{A}\cdot\boldsymbol{B}$

   A) *20.791* Nm
   B) *35.38* Nm
   C) *29.85* Nm
   D) *2.434* Nm
   E) *7.984* Nm

7. Given the vectors
   $\boldsymbol{A} = -2.47\,\boldsymbol{i} + 4.7\,\boldsymbol{j}$
   $\boldsymbol{B} = -8.33\,\boldsymbol{i} - 8.3\,\boldsymbol{j}$
   Calculate $\boldsymbol{A} \cdot \boldsymbol{B}$.
   A) *-18.435*
   B) *-22.861*
   C) *-13.879*
   D) *-26.536*
   E) *-11.776*

8. Given the vectors
   $\boldsymbol{A} = -7.4\,\boldsymbol{i} + 8.3\,\boldsymbol{j}$
   $\boldsymbol{B} = 9.1\,\boldsymbol{i} - 4.7\,\boldsymbol{j} + 1.5\,\boldsymbol{k}$
   Calculate the angle formed between the vectors.
   A) *157.509°*
   B) *131.285°*
   C) *69.688°*
   D) *203.947°*
   E) *176.259°*

9. Given the vectors
   $\boldsymbol{A} = 3.1$ N east and
   $\boldsymbol{B} = 1.5$ m $30°$ south of west,
   Calculate the magnitude of $\boldsymbol{A} \times \boldsymbol{B}$
   A) *1.739* Nm
   B) *2.325* Nm
   C) *3.074* Nm
   D) *1.071* Nm
   E) *3.958* Nm

10. Given the vectors
   $\boldsymbol{A} = \boldsymbol{k}$ and $\boldsymbol{B} = \boldsymbol{j}$,
   then $\boldsymbol{A} \times \boldsymbol{B}$ is equal to
   A) $-\boldsymbol{k}$
   B) $\boldsymbol{j}$
   C) $\boldsymbol{i}$
   D) $-\boldsymbol{i}$
   E) $-\boldsymbol{j}$

11. Given the vectors

$\boldsymbol{A} = 3.1\ \boldsymbol{i} - 4.7\ \boldsymbol{j} + 4.37\ \boldsymbol{k}$

$\boldsymbol{B} = 6.25\ \boldsymbol{i} - 4.7\ \boldsymbol{j} - 9.1\ \boldsymbol{k}$

Calculate the y-component of $\boldsymbol{A}\times\boldsymbol{B}$.

A) *90.945*
B) *83.706*
C) *55.523*
D) *103.614*
E) *21.467*

12. Given the vectors

$\boldsymbol{A} = -6.2\ \boldsymbol{i} + 9.41\ \boldsymbol{j}$

$\boldsymbol{B} = 9.71\ \boldsymbol{i} - 5.8\ \boldsymbol{j} + 7.4\ \boldsymbol{k}$

Calculate the magnitude of $\boldsymbol{A}\times\boldsymbol{B}$.

A) *143.505*
B) *162.051*
C) *121.294*
D) *172.097*
E) *100.121*

4 MOTION IN TWO DIMENSIONS

Your goal for this chapter is to learn about the relationships among motion variables for motion in a plane.

Motion in two dimensions is motion in a in a plane.

4.1 TWO DIMENSIONAL MOTION VARIABLES

Position vector ($\vec{r}$)*:* of a particle is the vector whose tail is at the origin and whose head is at the location of a particle. The x and y components of a position vector are simply its x and y coordinates.

$$r_x = x$$

$$r_y = y$$

In terms of the $\hat{i}$ and $\hat{j}$ unit vectors, a position vector of a particle may be written as

$$\vec{r} = x\hat{i} + y\hat{j}$$

Displacement vector ($\Delta\vec{r}$): of a particle is defined to be the change in its position vector.

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

Where $\vec{r}_i\left(\vec{r}_f\right)$ is initial (final) position vector. Graphically a displacement vector is the vector whose tail is at the initial location of the particle and whose head is at the final location of the particle. In terms of the $\hat{i}$ and $\hat{j}$ unit vectors, the displacement vector may be written as

$$\Delta\vec{r} = \left(x_f - x_i\right)\hat{i} + \left(y_f - y_i\right)\hat{j} = \Delta x\hat{i} + \Delta y\hat{j}$$

Average velocity ($\bar{\vec{v}}$): is defined to be displacement per a unit time.

$$\bar{\vec{v}} = \frac{\Delta\vec{r}}{\Delta t} = \left(\frac{\Delta x}{\Delta t}\right)\hat{i} + \left(\frac{\Delta y}{\Delta t}\right)\hat{j}$$

And since $\bar{\vec{v}} = \bar{v}_x\hat{i} + \bar{v}_y\hat{j}$, $\bar{v}_x = \dfrac{\Delta x}{\Delta t}$ and $\bar{v}_y = \dfrac{\Delta y}{\Delta t}$.

Instantaneous velocity ($\vec{v}$): is velocity at a given instant of time or average velocity evaluated at a very small interval time.

$$\vec{v} = \lim_{\Delta x \to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

And since $\vec{v} = v_x\hat{i} + v_y\hat{j}$, $v_x = \dfrac{dx}{dt}$ and $v_y = \dfrac{dy}{dt}$.

Average Acceleration ($\bar{\vec{a}}$): is defined to be change in velocity per a unit time.

$$\bar{\vec{a}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j}$$

And since $\bar{\vec{a}} = \bar{a}_x\hat{i} + \bar{a}_y\hat{j}$, $\bar{a}_x = \dfrac{\Delta v_x}{\Delta t}$ and $\bar{a}_y = \dfrac{\Delta v_y}{\Delta t}$.

Instantaneous Acceleration ($\vec{a}$)*:* is acceleration at a given instant of time or average acceleration evaluated at a very small interval of time.

$$\vec{a} = \lim_{\Delta t \to 0}\left(\frac{\Delta\vec{v}}{\Delta t}\right) = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}$$

And since $\vec{a} = a_x\hat{i} + a_y\hat{j}$, $a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$ and $a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}$.

Example: The position vector of a particle varies with time according to the equation $\vec{r} = \left(a_1t^2 + a_2t - a_3\right)\hat{i} + \left(a_4t - a_5\right)\hat{j}$ where
$a_1 = 4$ m/s^2; $a_2 = 2$ m/s; $a_3 = 1$ m; $a_4 = 8$ m/s and $a_5 = 5$ m.

a) Determine the position vector at t = 2.
Solution:

$$\vec{r}\,|_{t=2} = \left(a_1t^2 + a_2t - a_3\right)|_{t=2\text{ s}}\,\hat{i} + \left(a_4t - a_5\right)|_{t=2\text{ s}}\,\hat{j} = \left(19\hat{i} + 11\hat{j}\right) \text{ m}$$

b) Calculate the displacement of the particle between t = 1 s and t = 4 s.
Solution:

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = \vec{r}\,|_{t=4} - \vec{r}\,|_{t=1}$$

$$\vec{r}\big|_{t=4\text{ s}} = (4(4)^2 + 2(4) - 1)\hat{i}\text{ m} + (8(4) - 5)\hat{j}\text{ m} = 71\hat{i}\text{ m} + 27\hat{j}\text{ m}$$

$$\vec{r}\big|_{t=1\text{ s}} = \left(4(1)^2 + 2(1) - 1\right)\hat{i}\text{ m} + \left(8(1) - 5\right)\hat{j}\text{ m} = 5\hat{i}\text{ m} + 3\hat{j}\text{ m}$$

$$\Delta\vec{r} = \vec{r}\,|_{t=4\text{ s}} - \vec{r}\,|_{t=1\text{ s}} = \left(71\hat{i} + 27\hat{j}\right)\text{ m} - \left(5\hat{i} + 3\hat{j}\right)\text{ m} = 66\hat{i}\text{ m} + 24\hat{j}\text{ m}$$

c) Determine its average velocity between t = 0 and t = 5 s.
Solution:
$t_i = 0$; $t_f = 5$; $\overline{\vec{v}} = ?$

$$\vec{r}_f = \vec{r}\,|_{t=5\text{ s}} = \left(4\times5^2 + 2\times5 - 1\right)\hat{i}\text{ m} + \left(8\times5 - 5\right)\hat{j}\text{ m} = 109\hat{i}\text{ m} + 35\hat{j}\text{ m}$$

$$\vec{r}_i = \vec{r}\,|_{t=0} = \left(4\times0^2 + 2\times0 - 1\right)\hat{i}\text{ m} + \left(8\times0 - 5\right)\hat{j}\text{ m} = \left(-\hat{i} - 5\hat{j}\right)\text{ m}$$

$$\overline{\vec{v}} = \frac{\vec{r}_f - \vec{r}_i}{t_f - t_i} = \frac{\left(109\hat{i} + 35\hat{j}\right)\text{ m} - \left(-\hat{i} - 5\hat{j}\right)\text{ m}}{5\text{ s} - 0} = \frac{110\hat{i} + 40\hat{j}}{5}\text{ m/s} = \left(22\hat{i} + 8\hat{j}\right)\text{ m/s}$$

d) Calculate its instantaneous velocity at $t = 10$ s.
Solution:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(a_1t^2 + a_2t - a_3\right)\hat{i} + \frac{d}{dt}\left(a_4t - a_5\right)\hat{j} = \left(a_1t + a_2\right)\hat{i} + a_4\hat{j}$$

$$\vec{v}\big|_{t=10\text{ s}} = \left(8\times10 + 2\right)\hat{i}\text{ m/s} + 8\hat{j}\text{ m/s} = \left(82\hat{i} + 8\hat{j}\right)\text{ m/s}$$

e) Calculate its average acceleration between t = 2 s and t = 4 s.
Solution:

$$\vec{v}_f = \vec{v}\,\big|_{t=4\text{ s}} = \left(8\times 4+2\right)\hat{i}\ \text{m/s} + 8\hat{j}\ \text{m/s} = \left(34\hat{i}+8\hat{j}\right)\ \text{m/s}$$

$$\vec{v}_i = \vec{v}\,\big|_{t=2\text{ s}} = \left(8\times 2+2\right)\hat{i}\ \text{m/s} + 8\hat{j}\ \text{m/s} = \left(18\hat{i}+8\hat{j}\right)\ \text{m/s}$$

$$\bar{\vec{a}} = \frac{\vec{v}_f-\vec{v}_i}{t_f-t_i} = \frac{\left(34\hat{i}+8\hat{j}\right)-\left(18\hat{i}+8\hat{j}\right)}{4-2}\ \text{m/s} = \frac{16\hat{i}}{2}\ \text{m/s} = 8\hat{i}\ \text{m/s}$$

f) Calculate its instantaneous acceleration at t = 8 s.
Solution:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left(\left(a_1 t + a_2\right)\hat{i} + a_4\hat{j}\right)}{dt} = a_1\hat{i}$$

$$\vec{a}\big|_{t=8\text{ s}} = 8\hat{i}\ \text{m/s}^2$$

4.2 UNIFORMLY ACCELERATED MOTION

Uniformly accelerated motion is motion with constant acceleration. That is,

$$a_x = \text{ constant}$$

$$a_y = \text{ constant}$$

The horizontal and vertical components of a two dimensional motion are independent of each other in a sense that acceleration in one does not affect motion in the other. Thus the x-component and the y-component of the motion can be treated independently. In other words, a two dimensional motion can be treated as two one dimensional motions along the x-axis and the y-axis. The equations of a one dimensional uniformly accelerated motion have already been developed in an earlier chapter. Therefore the equations for the x-component and the y-component of a uniformly accelerated two dimensional motion can be given in the following two columns respectively as

$$\begin{array}{ll} v_{fx} = v_{ix} + a_x t & v_{fy} = v_{iy} + a_y t \\ \Delta x = v_{ix} t + \frac{1}{2} a_x t^2 & \Delta y = v_{iy} t + \frac{1}{2} a_y t^2 \\ \Delta x = \left(\frac{v_{ix} + v_{fx}}{2} \right) t & \Delta \text{y} = \left(\frac{v_{iy} + v_{fy}}{2} \right) t \\ v_{fx}^2 = v_{ix}^2 + 2 a_x \Delta \text{x} & v_{fy}^2 = v_{iy}^2 + 2 a_y \Delta \text{y} \end{array}$$

Only the first two of each column are independent. The others are dependent equations included for simplicity. The first two equations can also be written in vector form as follows:

$$\vec{v}_f = v_{fx}\hat{i} + v_{fy}\hat{j} = \left(v_{ix} + a_x t\right)\hat{i} + \left(v_{fy} + a_y t\right)\hat{j} = \left(v_{ix}\hat{i} + v_{iy}\hat{j}\right) + \left(a_x\hat{i} + a_y\hat{j}\right)t = \vec{v}_i + \vec{a}t$$

$$\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} = \left(v_{ix}t + \frac{1}{2}a_x t^2\right)\hat{i} + \left(v_{iy}t + \frac{1}{2}a_y t^2\right)\hat{j} = \left(v_{ix}\hat{i} + v_{iy}\hat{j}\right)t + \frac{1}{2}\left(a_x\hat{i} + a_y\hat{j}\right)t^2 = \vec{v}_i t + \frac{1}{2}\vec{a}t^2$$

Example: A particle is moving with a uniform acceleration of $(2\hat{i} + 4\hat{j})$ m/s^2. Its initial velocity is $\left(10\hat{i} + 5\hat{j}\right)$ m/s.

a) Determine its velocity after 4 seconds.
Solution:

$$\vec{v}_i = v_{ix}\hat{i} + v_{iy}\hat{j} = \left(10\hat{i} + 5\hat{j}\right) \text{ m/s; } \vec{a} = a_x\hat{i} + a_y\hat{j} = \left(2\hat{i} + 4\hat{j}\right) \text{ m/s}^2\text{; } t = 4 \text{ s; } \vec{v}_f = ?$$

$$\vec{v}_f = v_{fx}\hat{i} + v_{fy}\hat{j} = \left(v_{ix} + a_x t\right)\hat{i} + \left(v_{iy} + a_y t\right) = \left(10 + 2\times 4\right)\hat{i} \text{ m/s} + \left(10 + 4\times 4\right)\hat{j} \text{ m/s} = \left(18\hat{i} + 26\hat{j}\right) \text{ m/s}$$

b) Determine its displacement after 10 s.

Solution:

$\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j} = ?$

$$\Delta x = v_{ix}t + \frac{1}{2}a_x t^2 = \left[(10)(10) + \frac{1}{2}(2)(10)^2\right] \text{ m} = 200 \text{ m}$$

$$\Delta y = v_{iy}t + \frac{1}{2}a_y t^2 = \left[(5)(10) + \frac{1}{2}(4)(10)^2\right] \text{ m} = 250 \text{ m}$$

$$\Delta\vec{r} = \left(200\hat{i} + 250\hat{j}\right) \text{ m}$$

Practice Quiz 4.1

Choose the best answer

1. Which of the following statements is not correct?
   A) Average velocity and speed are the same.
   B) The x component of average velocity of a particle is equal to the change in its x coordinate per a unit time.
   C) Average velocity is change in displacement per a unit time.
   D) Instantaneous velocity is equal to average velocity evaluated at a very small interval time.
   E) Average velocity has the same direction as displacement.

2. A particle is displaced from the point (*-6* m, *-40* m) to the point (*-7* m, *-90* m). Calculate the vertical component of its displacement vector.
   A) *50* m
   B) *-1* m
   C) *1* m
   D) *-50* m
   E) *-130* m

3. A particle is displaced from the point (*4* m, *40* m) to the point (*7* m, *50* m) in *30* seconds. Calculate the vertical component of its average velocity.
   A) *0.226* m / s
   B) *0.333* m / s
   C) *0.057* m / s
   D) *0.51* m / s
   E) *0.548* m / s

4. The position vector of a certain particle varies with time according to the formula $\boldsymbol{r} = 9.3t^2\ \boldsymbol{i} + 6.8t\ \boldsymbol{j}$ Calculate the magnitude of its position vector after 7.1 seconds.
   A) 260.082 m
   B) 631.878 m
   C) 631.878 m
   D) 471.292 m
   E) 889.907 m

5. The position vector of a certain particle varies with time according to the formula $\boldsymbol{r} = 5.1t\ \boldsymbol{i} + 8.8t^2\ \boldsymbol{j}$ Calculate the direction of its displacement vector between $t = 8.3$ and $t = 18.3$ seconds.
   A) 59.747 °
   B) 144.867 °
   C) 164.29 °
   D) 88.752 °
   E) 112.649 °

6. The position vector of a certain particle varies with time according to the formula $\boldsymbol{r} = (8.5t + 7.4)\ \boldsymbol{i} + (6.8t + 9.7)\ \boldsymbol{j}$ Calculate the magnitude of its average velocity vector between $t = 8.3$ seconds and $t = 14.2$ seconds.
   A) 3.404 m / s
   B) 20.371 m / s
   C) 10.885 m / s
   D) 6.625 m / s
   E) 14.484 m / s

7. The position vector of a certain particle varies with time according to the formula $\boldsymbol{r} = 6.2t^2\ \boldsymbol{i} + 2.8t^3\ \boldsymbol{j}$ Calculate the magnitude of its instantaneous velocity after $t = 3.3$ seconds.
   A) 61.771 m / s
   B) 112.344 m / s
   C) 173.013 m / s
   D) 23.525 m / s
   E) 100.211 m / s

8. The position vector of a certain particle varies with time according to the formula $\boldsymbol{r} = 8.5t\ \boldsymbol{i} + 2.8t^2\ \boldsymbol{j}$ Calculate the magnitude of its average acceleration vector between $t = 4.2$ and $t = 14.2$ seconds.
   A) 4.705 m / s^2
   B) 0.919 m / s^2
   C) 5.664 m / s^2
   D) 1.549 m / s^2
   E) 2.466 m / s^2

9. The position vector of a certain particle varies with time according to the formula $\boldsymbol{r} = 4.8t\ \boldsymbol{i} + 2.8t^2\ \boldsymbol{j}$ Calculate the magnitude of its instantaneous acceleration after $t = 9.8$ seconds.
   A) 5.6 m / s^2
   B) 7.74 m / s^2
   C) 10.075 m / s^2
   D) 3.717 m / s^2
   E) 2.087 m / s^2

10. The velocity vector of a certain particle varies with time according to the formula $\boldsymbol{r} = (7.7t + 8.2)\ \boldsymbol{i} + (4.2t + 9.7)\ \boldsymbol{j}$ Calculate the magnitude of its instantaneous acceleration after $t = 4.2$ seconds.
    A) *13.831* m / s²
    B) *4.693* m / s²
    C) *5.974* m / s²
    D) *10.035* m / s²
    E) *8.771* m / s²

11. A particle is moving with a uniform acceleration of $(7.7\ \boldsymbol{i} + 8.3\ \boldsymbol{j})$ m / s². If its initial velocity is $(3.4\ \boldsymbol{i} + 8.3\ \boldsymbol{j})$ m / s, calculate the x-component of its displacement after *4.2* seconds.
    A) *131.305* m
    B) *82.194* m
    C) *141.212* m
    D) *123.152* m
    E) *90.333* m

4.3 PROJECTILE MOTION

Projectile motion is motion under gravity in a plane. Motion under gravity is a uniformly accelerated motion; that is, gravitational acceleration is a constant with a numerical value of 9.8 m/s². Its direction is downward. Thus its x-component is zero and its y-component is –9.8 m/s². Therefore for gravitational motion $a_x = 0$ and $a_y = -9.8\ \text{m/s}^2 = g$.

$$\vec{a} = g\,\hat{j} = \left(-9.8\ \frac{\text{m}}{\text{s}^2}\right)\hat{j}$$

Equations for a projectile motion can be obtained from the equations of a uniformly accelerated motion with $a_x = 0$ and $a_y = -9.8\ \text{m/s}^2 = g$.

$$v_{fx} = v_{ix} \qquad v_{fy} = v_{iy} + gt$$

$$\Delta x = v_{ix} t \qquad \Delta y = v_{iy} t + \frac{1}{2} g t^2$$

$$v_{fy}^{\ 2} = v_{iy}^{\ 2} + 2g\Delta y$$

$$\Delta y = \left(\frac{v_{iy} + v_{fy}}{2}\right) t$$

Example: A bullet is fired from a 10 m building horizontally with a speed of 1000 m/s.

a) Calculate the time taken to reach the ground.
Solution: Δy is negative because the final location is below the initial location.
$v_i = 1000$ m/s; $\theta_i = 0$; $\Delta y = -10$ m; $t = ?$

$$v_{iy} = 1000\sin(0) = 0$$

$$\Delta y = v_{iy}t + \frac{1}{2}gt^2$$

$$-10\text{ m} = \frac{1}{2}\left(-9.8\text{ m/s}^2\right)t^2$$

$$t = \sqrt{\frac{20}{9.8}}\text{ s} = 1.43\text{ s}$$

b) Calculate the x and y components of its velocity by the time it hits the ground.
Solution:

$$v_{fx} = v_{ix} = v_i\cos(\theta_i) = 1000\cos(0)\text{m/s} = 1000\text{ m/s}$$

$$v_{fy} = v_{iy} + gt = 0(1,43) - 9.8(1.43)\text{ m/s} = -14\text{ m/s}$$

c) How far did it fall horizontally?
Solution:

$$\Delta x = v_{ix} t = 1000(1.43)\ \text{m} = 1430\ \text{m}$$

d) Express its displacement and velocity by the time it hits the ground in terms of the $\hat{i}$ and $\hat{j}$ unit vectors.
Solution:

$$\Delta \vec{r} = \Delta x \hat{i} + \Delta y \hat{j} = \left(1430 \hat{i} - 10 \hat{j}\right)\ \text{m}$$

$$\vec{v}_f = v_{fx} \hat{i} + v_{fy} \hat{j} = \left(1000 \hat{i} - 14 \hat{j}\right)\ \text{m/s}$$

Example: A bullet is fired from the ground making an angle of with the horizontal with a speed of 1000 m/s.

a) How high will it rise?

Solution: The vertical component of the velocity at the maximum height is zero because the velocity is horizontal.
$v_i = 1000$ m/s; $\theta_i = 37°$; $v_{fy} = 0$; $\Delta y_{\max} = ?$

$$v_{iy} = v_i \sin(37) = 1000 \sin(37)\ \text{m/s} = 600\ \text{m/s}$$

$$v_{fy}{}^2 = v_{iy}{}^2 + 2g\Delta y$$

$$0 = (600\ \text{m/s})^2 + 2\left(-9.8\ \text{m/s}^2\right)$$

b) How long does it take to reach its maximum height?
Solution:

$$v_{fy} = v_{iy} + g\, t_{\max}$$

$$0 = (600\ \text{m/s}) + (-9.8\ \text{m/s})t_{\max}$$

$$t_{\max} = 61.2\ \text{s}$$

c) How far did it fall?

Solution: $\Delta y = 0$ because the initial and the final point are at the same level.

$\Delta y = 0$; $\Delta x = ?$

$$\Delta y = v_{iy} t + \frac{1}{2} g t^2$$

$$0 = (600 \text{ m/s}) t + \frac{1}{2}(-9.8 \text{ m/s}^2) t^2$$

$$t = 122.4 \text{ s}$$

$$v_{ix} = v_i \cos(\theta_i) = 1000 \cos(37°) \text{ m/s} = 800 \text{ m/s}$$

$$\Delta x = v_{ix} t = 800 \times 122.4 \text{ m/s} = 97920 \text{ m/s}$$

Example: A bullet is fired from a 100 m tall building with a speed of 1000 m/s making an angle of 53° with the horizontal right. Calculate the magnitude and direction of its velocity by the time it hits the ground.

Solution:

$\Delta y = -100$ m; $v_i = 1000$ m/s; $\theta_i = 53°$; $v_f = ?$; $\theta_f = ?$

$$v_{fx} = v_{ix} = v_i \cos(\theta_i) = 1000 \cos(53°) \text{ m/s} = 600 \text{ m/s}$$

$$v_{fy}{}^2 = v_{iy}{}^2 + 2g\Delta y = 800^2 + 2(-9.8)(-100) = 6.42 \times 10^5 \text{ m}^2/\text{s}^2$$

$$v_{fy} = -801.2 \text{ m/s}$$

$$v_f = \sqrt{v_{fx}{}^2 + v_{fy}{}^2} = \sqrt{600^2 + (-801.2)^2} \text{ m/s} = 3.6 \times 10^5 \text{ m/s}$$

$$\theta_f = \tan^{-1}\left(\frac{v_{fy}}{v_{fx}}\right) = \tan^{-1}\left(\frac{-801.2}{600}\right) = -53.2°$$

4.4 UNIFORM CIRCULAR MOTION

Uniform Circular Motion is motion in a circular path with a constant speed. The time taken for one complete revolution is called period (T). The number of cycles executed per second is called frequency (f). Period and frequency are inverses of each other.

$$f = \frac{1}{T}$$

The unit of measurement for frequency is Hertz, abbreviated as Hz. The number of radians executed per second is called angular speed (ω). Since there are 2π radians in one revolution,

$$\omega = 2\pi f = \frac{2\pi}{T}$$

The period of a circular motion can be obtained as a ratio between circumference and speed (v).

$$T = \frac{2\pi r}{v}$$

Where r stands for the radius of a circle.

Uniform Circular motion is an accelerated motion (even though the speed is constant) because direction is changing constantly. Acceleration due to change in direction is called centripetal or radial acceleration. As will be shown later, the direction of centripetal is always towards its center; and its magnitude (a_c) is proportional the square of the speed and inversely proportional to the radius of the circular path.

$$a_c = \frac{v^2}{r}$$

Example: A particle is revolving in a circular path of radius 4 m with a constant speed. If it makes 10 revolutions in 10 seconds, calculate its centripetal acceleration.

Solution: Let the number of revolutions be denoted by *N*.
$r = 4$ m; $t = 10$ s; $N = 10$; $a_c = ?$

$$T = \frac{t}{N} = \frac{10}{10} \text{ s} = 1 \text{ s}$$
$$v = \frac{2\pi r}{T} = \frac{2\pi \times 4}{1} \text{ m/s} = 25.1 \text{ m/s}$$
$$a_c = \frac{v^2}{r} = \frac{25.1^2}{4} = 157.5 \text{ m/s}^2$$

4.5 NON UNIFORM CIRCULAR MOTION

Non uniform circular motion is a circular motion where the speed is not constant. In this case, there are two kinds of accelerations. The acceleration due to change of direction which is called centripetal acceleration, and the acceleration due to change of speed which is called tangential acceleration (a_t). The direction of tangential acceleration is always tangent to the circular path, while the direction of centripetal acceleration is radial or towards the center. Therefor centripetal and tangential acceleration are perpendicular to each other. Thus the magnitude of the net acceleration (a_{net}) can be obtained from Pythagorean Theorem.

$$a_{net} = \sqrt{a_c^2 + a_t^2}$$

Example: The speed of a particle travelling in a circular path of radius 2 m changed uniformly from 5 m/s to 10 m/s in 10 seconds.

a) Calculate its tangential acceleration.
   Solution:
   $v_i = 5$ m/s; $v_f = 10$ m/s; $t = 10$ s; $a_t = ?$

$$a_t = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t} = \frac{10-5}{10} \text{ m/s}^2 = 0.5 \text{ m/s}^2$$

b) Calculate its initial centripetal acceleration.
   Solution:
   $r = 2$ m; $a_{ci} = ?$

$$a_{ci} = \frac{v_i^2}{r} = \frac{5^2}{2} \text{ m/s}^2 = 12.5 \text{ m/s}^2$$

c) Calculate its initial net acceleration.
Solution:

$$a_{net} = \sqrt{a_{ci}^2 + a_t^2} = \sqrt{12.5^2 + .5^2} \approx 12.5 \text{ m/s}^2$$

4.6 RELATIVE VELOCITY

Let $\vec{r}_{CA}$ be the position vector of particle C with respect to coordinate system A. Let $\vec{r}_{CB}$ be the position vector of particle C with respect to coordinate system B. And let $\vec{r}_{BA}$ be the position vector of coordinate system B with respect to coordinate system A. Then from vector addition, $\vec{r}_{CA} = \vec{r}_{CB} + \vec{r}_{BA}$. Taking the derivative of this equation $\frac{d\vec{r}_{CA}}{dt} = \frac{d\vec{r}_{CB}}{dt} + \frac{d\vec{r}_{BC}}{dt}$ or

$$\vec{v}_{CA} = \vec{v}_{CB} + \vec{v}_{BA}$$

$\vec{v}_{CA}$, $\vec{v}_{CB}$ and $\vec{v}_{BC}$ are the velocity of C with respect to A, the velocity of C with respect to C and the velocity of B with respect to A respectively.

Example: Car A is moving to the right with a speed of 20 m/s. Car B is traveling to the right with a speed of 50 m/s. Calculate the speed of car B with respect to car A.

Solution: Let the velocity of car B with respect to the ground, the velocity of car A with respect to the ground, the velocity of car B with respect to car A be denoted by $\vec{v}_{Bg}$, $\vec{v}_{Ag}$ and $\vec{v}_{BA}$ respectively.

Solution:
$\vec{v}_{Ag} = 20\hat{i}$ m/s; $\vec{v}_{Bg} = 50\hat{i}$ m/s; $\vec{v}_{BA} = ?$

$$\vec{v}_{Bg} = \vec{v}_{BA} + \vec{v}_{Ag}$$

$$\vec{v}_{BA} = \vec{v}_{Bg} - \vec{v}_{Ag} = \left(50\hat{i} - 20\hat{i}\right) \text{ m/s} = 30\hat{i} \text{ m/s}$$

Example: Car A is travelling with a speed of 50 m/s in a direction of north of east. Car B is travelling with a speed of 40 m/s in a direction of west.

a) Determine the velocity of car B with respect to car A.

Solution: Let the velocity of car B with respect to the ground, the velocity of car A with respect to the ground, the velocity of car B with respect to car A be denoted by $\vec{v}_{Bg}$, $\vec{v}_{Ag}$ and $\vec{v}_{BA}$ respectively.

$v_{Ag} = 50$ m/s; $\theta_{Ag} = 37°$; $v_{Bg} = 40$ m/s; $\theta_{Bg} = 180°$; $\vec{v}_{BA} = ?$

$$\vec{v}_{Ag} = v_{Ag}\cos\left(\theta_{Ag}\right)\hat{i} + v_{Ag}\sin\left(\theta_{Ag}\right) = \left(40\hat{i} + 30\hat{j}\right) \text{ m/s}$$

$$\vec{v}_{Bg} = v_{Bg}\cos\left(\theta_{Bg}\right)\hat{i} + v_{Bg}\sin\left(\theta_{Bg}\right) = -40\hat{i} \text{ m/s}$$

$$\vec{v}_{Bg} = \vec{v}_{BA} + \vec{v}_{Ag}$$

$$\vec{v}_{BA} = \vec{v}_{Bg} - \vec{v}_{Ag} = \left\{\left(-40\hat{i}\right) - \left(40\hat{i} + 30\hat{j}\right)\right\} \text{ m/s} = \left(-80\hat{i} - 30\hat{j}\right) \text{ m/s}$$

b) Determine the velocity of car A with respect to car B.

Solution:

$\vec{v}_{AB} = ?$

$$\vec{v}_{AB} = -\vec{v}_{BA} = -\left(-80\hat{i} - 40\hat{j}\right) \text{ m/s} = \left(80\hat{i} + 40\hat{j}\right) \text{ m/s}$$

Example: A river is flowing towards south with a speed of 5 m/s. A boat is travelling east with a speed of 8 m/s with respect to the river.

a) Determine the velocity of the boat with respect to the ground.
Solution: Let the velocity of boat with respect to the ground, the velocity of river with respect to the ground, the velocity of boat with respect to river be denoted by $\vec{v}_{bg}$, $\vec{v}_{rg}$ and $\vec{v}_{br}$ respectively.

$\vec{v}_{br} = 8\hat{i}$ m/s; $\vec{v}_{rg} = -5\hat{j}$ m/s; $\vec{v}_{bg} = ?$

$$\vec{v}_{bg} = \vec{v}_{br} + \vec{v}_{bg} = \left(8\hat{i} - 5\hat{j}\right) \text{ m/s}$$

b) How long with the boat take to cross the river if the width of the river is 40 m.
Solution: Let the width of the river be denoted by *d*. Let the component of the velocity across the river be denoted by $\vec{v}_d$ and let the time taken to cross the river be denoted by t_d
$d = 40$ m; $t_d = ?$

$$\vec{v}_d = \vec{v}_{bd} \cdot \hat{i} = \left(8\hat{i} - 5\hat{j}\right) \cdot \hat{i} \text{ m/s} = 8\hat{i} \text{ m/s}$$

$$t_d = \frac{d}{v_d} = \frac{40}{8} \text{ s} = 5 \text{ s}$$

Practice Quiz 4.2

Choose the best answer

1. Which of the following is a correct statement?
   A) The direction of the acceleration of a uniform circular motion is always directed towards the center.
   B) A uniform circular motion is a motion in a circular path with a constant acceleration.
   C) For a non-uniform circular motion, the acceleration due to change in direction and acceleration due to change in speed are parallel to each other
   D) For a particle moving in a circular path, the acceleration due to change in direction is tangent to the trajectory of the particle
   E) A uniformly accelerated circular motion is a motion in a circular path where the rate of change of acceleration is a constant

2. A bullet is fired horizontally from a *90* m tall building with a speed of *325* m / s. How long will it take to hit the ground?
   A) *6.854* s
   B) *4.286* s
   C) *1.026* s
   D) *7.746* s
   E) *3.401* s

3. A bullet is fired horizontally to the right from a *20* m tall building with a speed of *350* m / s. Calculate the direction of its displacementjust before it hits the ground.
   A) *-1.62*°
   B) *-2.521*°
   C) *-2.095*°
   D) *-0.959*°
   E) *-1.925*°

4. A bullet is fired from the ground making an angle of *60* deg with the horizontal-right with a speed of *325* m / s. Calculate the vertical component of its velocity by the time it hits the ground.
   A) *-161.163* m / s
   B) *-281.458* m / s
   C) *-251.419* m / s
   D) *-69.169* m / s
   E) *-390.295* m / s

5. A bullet is fired from the ground making an angle of *60* deg with the horizontal-right with a speed of *300* m / s. Calculate the magnitude of its displacement vector by the time it reaches its maximum height.
   A) *1145.723* m
   B) *6247.53* m
   C) *7897.622* m
   D) *1871.27* m
   E) *5260.61* m

6. A bullet is fired upwards from a *70* m tall building with a speed of *325* m / s making an angle of *30* deg with the horizontal-right. Calculate the time taken to hit the ground.
   A) *19.876* s
   B) *57.996* s
   C) *48.091* s
   D) *8.468* s
   E) *33.589* s

7. A bullet is fired upwards from a *60* m tall building with a speed of *225* m / s making an angle of *40* deg with the horizontal-right. Calculate the direction of its velocity just before it hits the ground.
   A) *-63.508°*
   B) *-29.2°*
   C) *-15.999°*
   D) *-40.773°*
   E) *-21.773°*

8. If the speed of a particle revolving in a circular path is multiplied by a factor of *19.1*, then its centripetal acceleration will be multiplied by a factor of
   A) *543.997*
   B) *364.81*
   C) *314.615*
   D) *457.19*
   E) *232.994*

9. The speed of an object revolving in a circular path of radius *13.6* m changed uniformly from *6.9* m / s to *16.3* m / in *9.2* seconds. Calculate its tangential acceleration.
   A) *1.527* m/s^2
   B) *0.161* m/s^2
   C) *1.648* m/s^2
   D) *0.753* m/s^2
   E) *1.022* m/s^2

10. Car A is going east with a speed of *14* m / s. Car B is going west with a speed of *30* m / s. Determine the velocity of car A with respect to car B.
   A) None of the other choices are correct.
   B) *44* m / s east
   C) *44* m / s west
   D) *16* m / s east
   E) *16* m / s east

11. Car A is going east with a speed of *20* m / s.
   Car B is going north with a speed of *30* m / s.
   Determine the direction of the velocity of car A with respect to car B.
   A) None of the other choices are correct.
   B) *-56.31°*
   C) *-33.69°*
   D) *33.69°*
   E) *56.31°*

5 NEWTON'S LAWS OF MOTION

Your goal for this chapter is to learn about the relationships between force and motion.

Newton's laws of motion are laws that establish the relationship between force and motion. In layman terms, force is a push, a pull, a squeeze and so on. Scientifically, force is something that causes change of motion or change of shape. The SI unit of measurement force is the Newton abbreviated as N.

Newton's first law states that an object will remain at rest or continue to move in a straight line with a constant sped unless acted upon by a net force.

Newton's second law: states that the net force $\left(\vec{F}_{net}\right)$ acting on an object is directly proportional to the acceleration $\left(\vec{a}\right)$ produced. The constant of proportionality between the net force and the acceleration is defined to be the mass (*m*) of the object. Mass is a measure of the amount of matter an object has.

$$\vec{F}_{net} = m\vec{a}$$

Unit of mass is N/(m/s^2) which is defined to be the kilogram (kg). Newton's second law may be written in component form as

$$F_{net\,x} = m\,a_x$$
$$F_{net\,y} = m\,a_y$$

If there are a number of forces acting on an object, the net force is the vector sum of all the forces acting on the object.

$$\vec{F}_{net} = \sum\vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots$$

$$\vec{F}_{net\,x} = \sum\vec{F}_x = \vec{F}_{1x} + \vec{F}_{2x} + \ldots$$

$$\vec{F}_{net\,y} = \sum\vec{F}_y = \vec{F}_{1y} + \vec{F}_{2y} + \ldots$$

5.1 TYPES OF FORCES

Forces may be classified as contact and non-contact forces. Contact forces are forces exerted by objects which are in direct contact with the object; while non-contact forces are forces exerted by objects which are not in direct contact with the object. Examples of non-contact forces are gravitational, magnetic and electromagnetic forces. In this course only gravitational force will be dealt with. While contact forces are self-explanatory, one kind of contact force, surface force, requires special attention.

Gravitational Force: Any two objects in the universe exert gravitational forces on each other. The gravitational force exerted on an object by the massive object in its vicinity is called the weight $(\vec{w})$ of the object. For example, the weight of an object on the surface of earth is the gravitational force exerted by earth on the object. From Newton's second law, weight is equal to the product of mass and gravitational acceleration.

$$\vec{w} = m\vec{g}$$
$$w = m|g|$$

The direction of weight (gravitational acceleration) is downward. In the $\hat{i}-\hat{j}$ notation, weight may be written as $\vec{w}=-m|g|\hat{j}$. The weight of an object varies from planet to planet because it depends on the masses of the planets; while mass is the same everywhere because it is a measure of the amount of matter the object has.

Surface Forces are forces exerted between two surfaces in contact. Surface force can be decomposed into a force perpendicular to the surfaces (pressing force) which is called a normal force (N) and a force parallel to the surface which is called friction (f). Friction and normal force are directly proportional. The greater the pressing (Normal) force, the greater the force of friction.

$$f=\mu N$$

Where the constant of proportionality μ is called the coefficient of friction of the surfaces in contact. Force of friction depends on the kinds of materials in contact. But it does not depend on the area of contact surface or the relative sliding speed between the surfaces.

Experiment shows that the force required to just get an object sliding is greater than the force required to get it sliding once it has started sliding. The friction that exists when an object is just starting to slide is called static friction (f_s) and the friction that exists once the object has started is called kinetic friction (f_k); That is $f_s>f_k$. Thus, there are two kinds of coefficients of friction: static coefficient of friction (μ_s) defined as

$$f_s=\mu_s N$$

and kinetic coefficient of friction (μ_k) defined as

$$f_k=\mu_k N$$

Since static friction is greater than kinetic friction, coefficient of static friction is greater than kinetic coefficient of friction.

$$\mu_s>\mu_k$$

Newton's third law states that for any action there is an equal but opposite reaction. If object A exerts force $\left(\vec{F}_{BA}\right)$ on object B, then object B also will exert force $\left(\vec{F}_{AB}\right)$ on object A such that

$$\vec{F}_{AB}=-\vec{F}_{BA}$$

Action reaction forces act on different objects and thus they cannot cancel each other. An object cannot exert force on itself, but it can use the principle of action reaction to move itself. For example, if a person wants to move forward, he has to push on the ground backward so that the reaction force moves him forward. For a rocket to move upward it has to push on the gases downward so that the reaction force of the gas propels it upward.

5.2 SOLVING FORCE PROBLEMS

The most important task in solving force problems is identifying the forces acting on the object. These forces can be contact or non-contact forces. For an object on the surface of earth there is always force of gravity which is its weight. The direction of weight is always vertically downward towards the center of earth. To identify the contact forces examine all the objects in contact with the object to determine the force exerted. If the surface of the object is in contact with another surface, remember to include surface force which is usually decomposed into friction (parallel to the surface) and normal force (perpendicular to the surface).

5.3 STATICS

Statics is the study of objects in equilibrium. An object is said to be in equilibrium (translational) if it is either at rest or moving in a straight line with a constant speed. An object will be in translational equilibrium if the net force acting on it is zero.

$$\vec{F}_{net} = \sum F = 0$$

In component form, this condition may be written as

$$\vec{F}_{net\,x} = \sum F_x = 0$$
$$\vec{F}_{net\,y} = \sum F_y = 0$$

Example: A 10 kg object is sliding on a horizontal surface with uniform speed by means of a horizontal string. The coefficient of friction between the object and the ground is 0.2. Calculate the normal force, the force of friction and the tension in the string (T).

Solution: Since the object is moving with a uniform speed in a straight line, it's acceleration is zero ($\vec{a} = 0$). The forces acting on the object are weight $(\vec{w})$, force exerted by the string $(\vec{T})$, normal force $(\vec{N})$ and force of friction $(\vec{f})$.
$m = 10$ kg; $\mu = 0.2$

$$\vec{T} = T\hat{i} \Rightarrow T_x = T \text{ and } T_y = 0$$

$$\vec{w} = -m|g|\hat{j} = -10 \times 9.8\hat{j}\text{ N} \Rightarrow w_x = 0 \text{ and } w_y = -98\text{ N}$$

$$\vec{N} = N\hat{j} \Rightarrow N_x = 0 \text{ and } N_y = N$$

$$\vec{f} = -f\hat{i} \Rightarrow f_x = -f \text{ and } f_y = 0$$

$$\vec{a} = 0 \Rightarrow a_x = 0 \text{ and } a_y = 0$$

$$T_y + w_y + f_y + N_y = ma_y$$

$$-98\text{ N} + N = 0$$

$$N = 98\text{ N}$$

$$f = \mu N = 0.2 \times 9.8\text{ N} = 19.6\text{ N}$$

$$T_x + w_x + f_x + N_x = ma_x$$

$$T - f = 0$$

$$T = f = 19.6\text{ N}$$

Example: A 10 kg object is hanging from a ceiling by means of two strings. String 1 makes an angle of 37° at the ceiling with the horizontal left. String 2 makes an angle of 53° at the ceiling with the horizontal right. Calculate the tensions in the strings.

Solution: The 10 kg object is at rest. Therefore $\vec{a} = 0$. The forces acting on the object are its weight $(\vec{w})$, the force exerted by string 1 $(\vec{T}_1)$ and the force exerted by string 2 $(\vec{T}_2)$

$m = 10$ kg; $\theta_1 = 37°$; $\theta_2 = 180 - 53 = 127°$; $T_1 = ?$; $T_2 = ?$

$$\vec{w} = -m|g|\hat{j} = -10 \times 9.8\hat{j} \text{ N} \Rightarrow w_x = 0 \text{ and } w_y = -98 \text{ N}$$

$$\vec{T}_1 = T_1 \cos(37°)\hat{i} + T_1 \sin(37°) \Rightarrow T_{1x} = 0.8T_1 \text{ and } T_{1y} = 0.6T_1$$

$$\vec{T}_2 = T_2 \cos(127°)\hat{i} + T_2 \sin(127°) \Rightarrow T_{2x} = -0.6T_2 \text{ and } T_{2y} = 0.8T_2$$

$$\vec{a} = 0 \Rightarrow a_x = 0 \text{ and } a_y = 0$$

$$w_x + T_{1x} + T_{2x} = ma_x$$

$$0.8T_1 - 0.6T_2 = 0$$

$$T_1 = \frac{3}{4}T_2$$

$$w_y + T_{1y} + T_{2y} = ma_y$$

$$-98 \text{ N} + 0.6T_1 + 0.8T_2 = 0$$

$$0.6T_1 + 0.8T_2 = -98 \text{ N}$$

$$0.6T_1 + 0.8T_2 = -98 \text{ N and } T_1 = \frac{3}{4}T_2 \Rightarrow T_1 = 60 \text{ N and } T_2 = 80 \text{ N}$$

Practice Quiz 5.1

Choose the best answer

1. Newton's third law states that
   A) Any two objects in the universe attract each other with a force directly proportional to the product of the masses and inversely proportional to the square of the distance separating them.
   B) An object will remain at rest or move in a straight line unless acted upon by a net force.
   C) For every reaction, there is an equal reaction in the same direction.
   D) For every action, there is an equal but opposite reaction.
   E) The force acting on an object is directly proportional to the acceleration produced by the force.

2. Which of the following situations can be explained by Newton's third law?
    A) A feather falling in a straight line with a constant speed.
    B) When the force acting on an object is doubled its acceleration is doubled.
    C) When a man fires a gun, his body jerks backwards.
    D) Two objects of different masses released from the same height at the same time hit the ground at the same time.
    E) When a driver applies the breaks suddenly, his body jerks forward

3. Weight of an object is
    A) A measure of the amount of matter an object has.
    B) The same on different planets.
    C) The same with the mass of the object.
    D) measured in terms of kilograms
    E) A measure of the gravitational force exerted on the object by the massive object in the vicinity of the object.

4. The component of the surface force between two surfaces sliding on each other perpendicular to the surface is called
    A) normal force
    B) contact force
    C) gravitational force
    D) friction
    E) weight

5. Which of the following is not a correct statement?
    A) Static coefficient of friction is greater than kinetic coefficient of friction
    B) Coefficient of friction is equal to the ratio between normal force and force of friction.
    C) The force required to just get an object sliding is bigger than the force required to slide it once it has started sliding.
    D) Coefficient of friction is unit less.
    E) If normal force is doubled, force of friction will be doubled.

6. When an object of mass *18* kilogram is acted upon by a certain force, it moves with an acceleration of $(6.4\,\boldsymbol{i} + 1.5\,\boldsymbol{j})$ m / s^2. Calculate the magnitude of the net force acting on the object.
   A) *77.557* N
   B) *218.196* N
   C) *52.488* N
   D) *118.322* N
   E) *133.98* N

7. Under the influence of a certain force, a particle of mass *16* kg initially moving with a speed of *11.5* m / s was brought to rest in a distance of *7.79* m. Calculate the force acting on the particle.
   A) *184.509* N
   B) *29.179* N
   C) *135.815* N
   D) *121.469* N
   E) *166.405* N

8. The forces ($3.6\, \boldsymbol{i} + 8.3\, \boldsymbol{j}$) N, ($1.4\, \boldsymbol{i} + 1.4\, \boldsymbol{j}$) N and an unknown force are acting on an object of mass 5 kg in equilibrium, Calculate the magnitude of the unknown force.
   A) 2.79 N
   B) 4.059 N
   C) 6.258 N
   D) 10.913 N
   E) 15.284 N

9. An object is being pulled on a horizontal surface with a horizontal force of 135 N. If it is moving with a uniform speed and the coefficient of friction between the sliding surfaces is 0.5, calculate the normal force exerted by the surface on the object.
   A) 135 N
   B) 350.68 N
   C) 67.5 N
   D) 270 N
   E) 0.006 N

10. An object of mass 700 kg is hanging from a ceiling by means of two strings. The first string (T_1) makes an angle of 20 degree with the horizontal-right. The second string (T_2) makes an angle of 25 degree with the horizontal-left. The equations relating the x-components and the y-components of the forces acting on the object respectively are
   A) $-0.94 * T_1 + 0.423 * T_2 = 0$
   $-0.94 * T_1 + 0.423 * T_2 - 6860$ N $= 0$
   B) $-0.94 * T_1 + 0.906 * T_2 - 6860$ N $= 0$
   $0.342 * T_1 + 0.423 * T_2 = 0$
   C) $-0.94 * T_1 + 0.906 * T_2 = 0$
   $0.342 * T_1 + 0.423 * T_2 - 6860$ N $= 0$
   D) $0.342 * T_1 + 0.423 * T_2 - 6860$ N $= 0$
   $-0.94 * T_1 + 0.906 * T_2 = 0$
   E) $0.342 * T_1 + 0.423 * T_2 = 0$
   $-0.94 * T_1 + 0.906 * T_2 - 6860$ N $= 0$

11. An object of mass *700* kg is hanging from a ceiling by means of two strings. The first string (T_1) makes an angle of *60* degree with the horizontal-right. The second string (T_2) makes an angle of *50* degree with the horizontal-left. Calculate the tension in the first string (T_1)
    A) *4692.516* N
    B) *3086.39* N
    C) *1072.877* N
    D) *6567.104* N
    E) *8041.748* N

5.4 DYNAMICS

Dynamics is the study of accelerated systems. From Newton's second law

$$\vec{F}_{net} = m\vec{a}$$

In component form,

$$\vec{F}_{net\,x} = m a_x$$
$$\vec{F}_{net\,y} = m a_y$$

Also since $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$,

$$\vec{F}_{net} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2}$$

In component form,

$$\vec{F}_{net\,x} = m\frac{dv_x}{dt} = m\frac{d^2x}{dt^2}$$
$$\vec{F}_{net\,y} = m\frac{dv_y}{dt} = m\frac{d^2y}{dt^2}$$

Example: The position vector of a certain particle of mass 0.2 kg varies with time according to the equation $\vec{r} = \left(a_1t^2 - a_2t\right)\hat{i} + \left(a_3t^3 + a_4\right)\hat{j}$ where $a_1 = 3$ m/s^2, $a_2 = 4$ m/s, $a_3 = 4$ m/s^3 and $a_4 = 8$ m. Calculate the force acting on the particle at $t = 2$ s.

Solution:

$$\vec{r} = \left(a_1t^2 - a_2t\right)\hat{i} + \left(a_3t^3 + a_4\right)\hat{j}$$

$$\frac{d\vec{r}}{dt} = \left(2a_1t - a_2\right)\hat{i} + \left(3a_3t^2\right)\hat{j}$$

$$\frac{d^2\vec{r}}{dt^2} = 2a_1\hat{i} + 6a_3t\hat{j}$$

$$\frac{d^2\vec{r}}{dt^2} = 6\hat{i} + 24t\hat{j}$$

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2}$$

$$\vec{F}\Big|_{t=2\text{ s}} = m\frac{d^2\vec{r}}{dt^2}\Bigg|_{t=2\text{ s}} = 0.2\left(6\hat{i} + 48\hat{j}\right)\text{ N} = \left(1.2\hat{i} + 9.6\hat{j}\right)\text{ N}$$

Example: A 4 kg object is being acted upon on by 3 forces. Two of the forces are equal to $\left(2\hat{i} + 3\hat{j}\right)$ and $\left(-4\hat{i} + 5\hat{j}\right)$ respectively. The third force is unknown. If the object is moving with an acceleration of $\left(3\hat{i} + 6\hat{j}\right)$ m/s^2 calculate the unknown force.

Solution: Let the two known forces and the unknown force be denoted by $\vec{F}_1, \vec{F}_2$ and $\vec{F}_3$ respectively.

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = m\vec{a}$$

$$\left(2\hat{i} + 3\hat{j}\right)\text{ N} + \left(-4\hat{i} + 5\hat{j}\right)\text{ N} + \vec{F}_3 = 4\left(3\hat{i} + 6\hat{j}\right)\text{ N}$$

$$\vec{F}_3 = \left(14\hat{i} + 16\hat{j}\right)\text{ N}$$

Example: A 10 kg object is being pulled horizontally by a force of 60 N.

a) Assuming no friction, calculate the acceleration of the object.
Solution: The forces acting are the horizontal force $(\vec{F})$, normal force $(\vec{N})$ and weight $(\vec{w})$. $\vec{N}$ and $\vec{w}$ do not contribute to the horizontal motion because they are vertical forces.
$m = 10$ kg; $\vec{F} = 60\hat{i}$ N; $a = ?$

$$\vec{F} = 60\hat{i}\ \text{N} \Rightarrow F_x = 60 \text{ and } F_y = 0$$
$$\vec{a} = a\hat{i} \Rightarrow a_x = a \text{ and } a_y = 0$$
$$F_x = ma_x$$
$$60\ \text{N} = (10\ \text{kg})a$$
$$a = 6\ \text{m/s}^2$$

b) Assuming the coefficient of friction between the surfaces is 0.2, calculate the acceleration of the object.
Solution: Forces Acting are the horizontal force $(\vec{F})$, normal force $(\vec{N})$, weight $(\vec{w})$ and friction $(\vec{f})$.

Solution:

$$\vec{N} = N\hat{j} \Rightarrow N_x = 0 \text{ and } N_y = N$$
$$\vec{w} = -m|g|\hat{j} \Rightarrow w_x = 0 \text{ and } w_y = -m|g|$$
$$\vec{f} = -f\hat{i} \Rightarrow f_x = -f \text{ and } f_y = 0$$
$$N_y + w_y + f_y = ma_y$$
$$N - mg = 0$$
$$N = m|g| = 10\times 9.8\ \text{N} = 98\ \text{N}$$
$$f = \mu N = 0.2\times 98\ \text{N} = 19.6\ \text{N}$$
$$N_x + F_x + f_x + w_x = ma_x$$
$$(60 - 19.6)\ \text{N} = (10\ \text{kg})a$$
$$a = 4.04\ \text{m/s}^2$$

Example: A 10kg object is being pulled horizontally by a 100 N force that makes an angle of with the horizontal right.

a) Assuming no friction, calculate its acceleration and the normal force.
Solution: Forces acting are pulling force $(\vec{F})$, normal force $(\vec{N})$ and weight $(\vec{w})$.
$F = 100\text{ N};\ \theta_F = 37°;\ m = 10\text{ kg};\ N = ?;\ a = ?$

$$\vec{F} = \left(100\cos(37°)\hat{\imath} + 100\sin(37°)\hat{\jmath}\right)\text{ N} \Rightarrow F_x = 80\text{ N and } F_y = 60\text{ N}$$

$$\vec{N} = N\hat{\jmath} \Rightarrow N_x = 0 \text{ and } N_y = N$$

$$\vec{w} = -m|g|\hat{\jmath} = -10 \times 9.8\hat{\jmath}\text{ N} \Rightarrow w_x = 0 \text{ and } w_y = -98\text{ N}$$

$$\vec{a} = a\hat{i} \Rightarrow a_x = a \text{ and } a_y = 0$$

$$F_y + N_y + w_y = ma_y$$

$$(60 - 98)\text{ N} + N = 0$$

$$N = 38\text{ N}$$

$$F_x + N_x + w_x = ma_x$$

$$80\text{ N} = (10\text{ kg})a$$

$$a = 8\text{ m/s}^2$$

b) Assuming the coefficient of friction between the surfaces is 0.2, calculate its acceleration.
Solution: Forces Acting are the horizontal force $(\vec{F})$, normal force $(\vec{N})$, weight $(\vec{w})$ and friction $(\vec{f})$.
$\mu = 0.2$

$$\vec{f} = -f\hat{\imath} = -\mu N\hat{\imath} = -0.2 \times 38\text{ N} \Rightarrow f_x = 0 \text{ and } f_y = -7.6\text{ N}$$

$$F_x + N_x + w_x + f_x = ma_x$$

$$(80 - 7.6)\text{ N} = 10a$$

$$a = 8.24\text{ m/s}^2$$

Example: Consider an object of mass m sliding down an inclined plane that makes an an angle θ with the horizontal left.

a) Assuming no friction, express the normal force and its acceleration in terms of m and θ
Solution: Forces acting are normal force $\left(\vec{N}\right)$ and weight $\left(\vec{w}\right)$. Let's use a coordinate system where the positive x-axis is parallel to the acceleration of the object. Then the y-axis is perpendicular to the inclined plane in a counterclockwise direction from the positive x-axis (perpendicularly upward). In this coordinate system the weight makes an angle of 90 – θ with the positive axis. Therefore $\theta_w = -\left(90-\theta\right)$ and the normal force is parallel to the positive y-axis (That is $\theta_N = 90°$)
$\vec{w} = m|g|\cos\left(\theta-90\right)\hat{i} + m|g|\sin\left(\theta-90\right)\hat{j} \Rightarrow w_x = m|g|\sin\left(\theta\right)$ and $w_y = -m|g|\cos\left(\theta\right)$

$$\vec{N} = N\hat{j} \Rightarrow N_x = 0 \text{ and } N_y = N$$

$$\vec{a} = a\hat{i} \Rightarrow a_x = a \text{ and } a_y = 0$$

$$w_y + N_y = ma_y$$

$$-m|g|\cos\left(\theta\right) + N = 0$$

$$N = m|g|\cos(\theta)$$

$$w_x + N_x = ma_x$$

$$m|g|\sin(\theta) = ma$$

$$a = |g|\sin(\theta)$$

b) Assuming the coefficient of friction between the surfaces is μ, express the normal force and the acceleration in terms of m and θ.
Solution: Forces Acting are normal force $(\vec{N})$, weight $(\vec{w})$ and friction $(\vec{f})$.
$\vec{w} = m|g|\cos(\theta - 90)\hat{i} + m|g|\sin(\theta - 90)\hat{j} \Rightarrow w_x = m|g|\sin(\theta)$ and $w_y = -m|g|\cos(\theta)$

$$\vec{N} = N\hat{j} \Rightarrow N_x = 0 \text{ and } N_y = N$$

$$\vec{f} = -f\hat{i} = -\mu N\hat{i} \Rightarrow f_x = -\mu N \text{ and } f_y = 0$$

$$\vec{a} = a\hat{i} \Rightarrow a_x = a \text{ and } a_y = 0$$

$$w_y + N_y + f_y = ma_y$$

$$-m|g|\cos(\theta) + N = 0$$

$$N = m|g|\cos(\theta)$$

$$w_x + N_x + f_x = ma_x$$

$$m|g|\sin(\theta) - \mu N = ma$$

$$N = m|g|\cos(\theta) \text{ and } m|g|\sin(\theta) - \mu N = ma \Rightarrow a = |g|(\sin(\theta) - \cos(\theta))$$

Example: An object of mass 5 kg on a table is attached to a hanging object of mass 10 kg by a string via a pulley. Assuming no friction, calculate the tension in the string and the acceleration of the objects.

Solution: Let the subscripts 1 and 2 represent the object on the table and the hanging objects respectively. Forces acting on the object on the table are tension in string $(\vec{T}_1)$, weight $(\vec{w}_1)$ and normal force $(\vec{N}_1)$. Forces acting on the hanging object are tension in string $(\vec{T}_2)$ and weight $(\vec{w}_2)$. The magnitude of the tension is the same throughout $(T_1 = T_2 = T)$. The magnitude of the acceleration is the same for both objects $(a_1 = a_2 = a)$.

Object on the table:

$$\vec{T}_1 = T\hat{i} \Rightarrow T_{1x} = T \text{ and } T_{1y} = 0$$

$$\vec{a}_1 = a\hat{i} \Rightarrow a_{1x} = a \text{ and } a_{1y} = 0$$

$$T_{1x} = m_1 a_{1x} = m_1 a$$

$$T = (5\text{ kg})a$$

Hanging object:

$$\vec{T}_2 = T\hat{j} \Rightarrow T_{2x} = 0 \text{ and } T_{2y} = T$$

$$\vec{w}_2 = -m_2|g|\hat{j} \Rightarrow w_{2x} = 0 \text{ and } w_{2y} = -m_2|g|$$

$$\vec{a}_2 = -a\hat{j} \Rightarrow a_{2x} = 0 \text{ and } a_{2y} = -a$$

$$T_{2y} + w_{2y} = m_2 a_{2y}$$

$$T - m_2|g| = -m_2 a$$

$$T + (10\text{ kg})a = 98\text{ N}$$

$$T = (5\text{ kg})a \text{ and } T + (10\text{ kg})a = 98\text{ N} \Rightarrow a = 6.53\text{ m/s}^2 \text{ and } T = 32.7\text{ N}$$

Example: Consider two objects of masses m_1 and m_2 attached by a string hanging from a pulley. (Assume the pulley is frictionless and weightless. Also assume $m_2 > m_1$).

Solution: Forces acting on m_1 are tension in the string $(\vec{T}_1)$ and its weight $(\vec{w}_1)$. Forces acting on m_2 are the tension in the string $(\vec{T}_2)$ and its weight $(\vec{w}_2)$. The magnitude of the tension is the same throughout the string $(T_1 = T_2 = T)$. The magnitude of the acceleration is the same for both objects $(a_1 = a_2 = a)$.

Object of mass m_1:

$$\vec{T}_1 = T\hat{j} \Rightarrow T_{1x} = 0 \text{ and } T_{1y} = T$$

$$\vec{w}_1 = -m_1|g|\hat{j} \Rightarrow w_{1x} = 0 \text{ and } w_{1y} = -m_1|g|$$

$$\vec{a}_2 = -a\hat{j} \Rightarrow a_{2x} = 0 \text{ and } a_{2y} = -a$$

$$T_{1y} + w_{1y} = m_1 a_{1y}$$

$$T - m_1|g| = m_1 a$$

Object of mass m_2:

$$\vec{T}_2 = T\hat{j} \Rightarrow T_{2x} = 0 \text{ and } T_{2y} = T$$

$$\vec{w}_2 = -m_2|g|\hat{j} \Rightarrow w_{2x} = 0 \text{ and } w_{2y} = -m_2|g|$$

$$\vec{a}_2 = -a\hat{j} \Rightarrow a_{2x} = 0 \text{ and } a_{2y} = -a$$

$$T_{2y} + w_{2y} = m_2 a_{2y}$$

$$T - m_2|g| = -m_2 a$$

$$T - m_1|g| = m_1 a \text{ and } T - m_2|g| = -m_2 a \Rightarrow a = \frac{(m_2 - m_1)|g|}{m_1 + m_2}$$

Practice Quiz 5.2

Choose the best answer

1. The position vector of a certain particle of mass *4.7* kg varies with time according to the equation $\boldsymbol{r} = 9.4t^3\,\boldsymbol{i} + 2.5t^2\,\boldsymbol{j}$ Calculate the direction of the force acting on the particle after *11.5* seconds.
   A) *0.207*°
   B) *0.146*°
   C) *0.442*°
   D) *0.056*°
   E) *0.631*°

2. The cable of an elevator is supporting a mass of *232.3* kg. Calculate the tension in the cable when the cable is moving upwards with an acceleration of *2.63* $\mathrm{m/s^2}$.
   A) *5165.936* N
   B) *1598.549* N
   C) *2887.489* N
   D) *4023.717* N
   E) *2562.543* N

3. An object of mass *10* kg is being pulled on a friction less horizontal surface by means of a string that makes an angle of *30* degree with the horizontal. If the tension in the string is *10* N calculate the acceleration of the object.
   A) *0.73* $\mathrm{m/s^2}$
   B) *1.343* $\mathrm{m/s^2}$
   C) *1.113* $\mathrm{m/s^2}$
   D) *0.866* $\mathrm{m/s^2}$
   E) *1.489* $\mathrm{m/s^2}$

4. An object of mass *20* kg is being pulled on a friction less horizontal surface by means of two strings. One of the strings has a tension of *18* N and makes an angle of *30* degree with the horizontal. The other string is pulling horizontally and has a tension of *2* N. Calculate the acceleration of the object.
   A) *0.879* m/s^2
   B) *1.449* m/s^2
   C) *1.345* m/s^2
   D) *0.238* m/s^2
   E) *1.146* m/s^2

5. An object of mass *36* kg is being pulled on a friction less horizontal surface by means of two strings. One of the strings is pulling forward, has a tension of *300* N and makes an angle of *40* degree with the horizontal-right. The other string is pulling backwards horizontally and has a tension of *12* N. Calculate the acceleration of the object.
   A) *3.696* m/s^2
   B) *0.761* m/s^2
   C) *5.433* m/s^2
   D) *7.127* m/s^2
   E) *6.05* m/s^2

6. An object of mass *3* kg is being pulled on a horizontal surface by a string that makes an angle of *60* degree with the horizontal. The tension in the string is *16* N. Calculate the normal force exerted by the ground on the object.
   A) *10.966* N
   B) *25.121* N
   C) *21.11* N
   D) *15.544* N
   E) *7.822* N

7. An object is of mass *48* kg is being pulled on a horizontal surface by a string that makes an angle of *15* deg with the horizontal. The tension in the string is *360* N. The coefficient of kinetic friction of the surfaces is *0.4*. Calculate the acceleration of the object.
   A) $5.686\ \mathrm{m/s^2}$
   B) $2.692\ \mathrm{m/s^2}$
   C) $4.101\ \mathrm{m/s^2}$
   D) $2.19\ \mathrm{m/s^2}$
   E) $0.548\ \mathrm{m/s^2}$

8. An object of mass *28* kg is sliding down a friction less inclined plane that makes an angle of *20* deg with the horizontal. Calculate its acceleration down the plane.
   A) $9.8\ \mathrm{m/s^2}$
   B) $93.85\ \mathrm{m/s^2}$
   C) $3.352\ \mathrm{m/s^2}$
   D) $257.852\ \mathrm{m/s^2}$
   E) $9.209\ \mathrm{m/s^2}$

9. An object of mass *2* kg is sliding in an inclined plane of inclination *30°*. The coefficient of kinetic friction between the surfaces of the object and the plane is *0.35*. Calculate its acceleration down the plane.
   A) *2.646* m/s^2
   B) *1.409* m/s^2
   C) *2.942* m/s^2
   D) *1.93* m/s^2
   E) *0.352* m/s^2

10. An object of mass *7.1* kg on a friction-less table is connected to a hanging object of mass *18.2* kg by means of a string via a pulley. Calculate the acceleration of the system.
   A) *3.269* m/s^2
   B) *10.403* m/s^2
   C) *7.05* m/s^2
   D) *3.979* m/s^2
   E) *11.453* m/s^2

11. An object of mass *5.2* kg on a table is connected to a hanging object of mass *15.2* kg by means of a string via a pulley. The coefficient of kinetic friction between the table and the sliding object is *0.3*. Calculate the acceleration of the system.
   A) *4.505* m/s^2
   B) *10.708* m/s^2
   C) *6.553* m/s^2
   D) *3.063* m/s^2
   E) *8.615* m/s^2

6 CIRCULAR MOTION AND APPLICATIONS OF NEWTON'S SECOND LAW

Your goal for this chapter is to learn about the relationship between force and motion variables for a circular motion.

It is easier to describe circular motion in terms of polar coordinates (r,θ) because even though it is a two-dimensional problem in terms of Cartesian coordinates (x,y), it is a one dimensional problem in terms of polar coordinates. In a polar coordinate system, a point is identified by its distance from the origin (r) and the angle formed between the position vector of the point and the positive x-axis measured in a counter clockwise direction (negative if measured clockwise). Relationships between Cartesian and polar coordinates can easily be obtained using the right angled triangle formed by the position vector of the point and the positive x-axis. Cartesian coordinates. Cartesian coordinates can be obtained from polar coordinates using the equations

$$x = r\cos\theta$$
$$y = r\sin\theta$$

And polar coordinates can be obtained from Cartesian coordinates from the equations

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \; (+180^\circ \text{ if } x < 0)$$

6.1 POLAR UNIT VECTORS

r-unit vector $(\hat{e}_r)$ has the same direction as the change of the position $(\vec{r})$ vector with respect to *r* while keeping θ constant. In other words $\hat{e}_r$ has the same direction as the partial derivative of the position vector with respect to *r*. Therefore $\hat{e}_r$ can be obtained from $\hat{e}_r = {\frac{\partial \vec{r}}{\partial r}}\Big/{\left|\frac{\partial \hat{r}}{\partial r}\right|}$. But

$\frac{\partial \vec{r}}{\partial r} = \frac{\partial}{\partial r}\left(r\cos(\theta)\hat{\imath} + r\sin(\theta)\hat{\jmath}\right) = \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}$ and $\left|\frac{\partial \vec{r}}{\partial r}\right| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$.
Therefore the r-unit vector is given as

$$\hat{e}_r = \cos(\theta)\hat{\imath} + \sin(\theta)\hat{\jmath}$$

Comparing with the expression for the position vector, $\vec{r} = r\left(\cos(\theta)\hat{\imath} + \sin(\theta)\, j\right)$, the r-unit vector may also be given as

$$\hat{e}_r = \frac{\vec{r}}{r}$$

The r-unit vector has the same direction as the position vector which is radially outward from the origin.

θ-*unit vector* $(\hat{e}_\theta)$ has the same direction as the change of the position vector with respect to θ while keeping r constant. In other words the θ-unit vector is parallel to the partial derivative of the position vector with respect to θ. Thus the θ-unit vector may be given as $\hat{e}_\theta = \frac{\partial\vec{r}}{\partial\theta} \Big/ \left|\frac{\partial\vec{r}}{\partial\theta}\right|$. But $\frac{\partial\vec{r}}{\partial\theta} = \frac{\partial}{\partial\theta}\left(r\cos(\theta)\hat{\imath} + r\sin(\theta)\hat{\jmath}\right) = r\left(-\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}\right)$ and $\left|\frac{\partial\vec{r}}{\partial r}\right| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$. Therefore the θ-unit vector is given as

$$\hat{e}_\theta = -\sin(\theta)\hat{\imath} + \cos(\theta)\hat{\jmath}$$

The dot product between $\hat{e}_r$ and $\hat{e}_\theta$ is zero which implies the θ-unit vector is perpendicular to the r-unit vector. And the fact that $\hat{e}_r \times \hat{e}_\theta = \hat{k}$ implies that $\hat{e}_\theta$ is perpendicular to $\hat{e}_r$ in a counter clockwise direction. In other words the θ-unit vector is tangent to a circle centered at the origin in a counter clockwise direction.

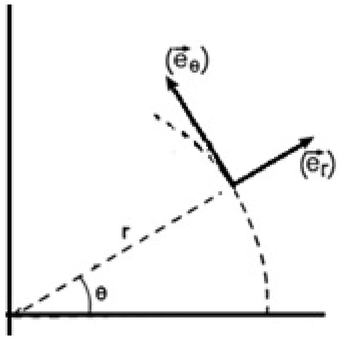

Figure 6.1

By direct differentiation, the following expression for the rate of change of the unit vectors can be obtained.

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$$

Example: A particle is located at the point $(3,4)$ m. Express its position vector in terms of polar unit vectors.

Solution:
$x = 3$ m; $y = 4$ m

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} \text{ m} = 5 \text{ m}$$

$$\vec{r} = r\hat{e}_r = 5\hat{e}_r \text{ m}$$

6.2 CIRCULAR MOTION IN TERMS OF POLAR COORDINATES

For a circular motion, if the origin is the center of the circular path, r remains constant. First let's express the motion variables in terms of polar unit vectors.

Position vector $(\vec{r})$*:* as shown earlier

$$\vec{r} = r\hat{e}_r$$

Velocity $(\vec{v})$: $\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = r\frac{d\hat{e}_r}{dt} = r\frac{d\hat{e}}{d\theta}\frac{d\theta}{dt}$. But $\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta$ and $\frac{d\theta}{dt}$ is the rate of change of angular position with respect time which is known as angular speed and denoted by ω. Thus $\vec{v} = r\omega\hat{e}_\theta$. Again $r\omega = r\frac{d\theta}{dt} = \frac{d}{dt}(r\theta)$ But $r\theta = s$ which is the arc length. Therefore $r\omega = \frac{ds}{dt}$ which is the linear speed v $(v = \omega r)$ and thus the velocity can be expressed in terms of polar unit vector as

$$\vec{v} = v_\theta\hat{e}_\theta$$

As expected the direction of the velocity is tangent to the circular trajectory. Since the direction of $\hat{e}_\theta$ is in a counter clockwise direction, v_θ is taken to be positive if the particle is moving in a counter clockwise direction and negative if it is moving in a clockwise direction.

Acceleration $(\vec{a})$: $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v_\theta\hat{e}_\theta) = v_\theta\frac{d\hat{e}_\theta}{dt} + \frac{dv_\theta}{dt}\hat{e}_\theta$ But $\frac{d\hat{e}_\theta}{dt} = \frac{d\hat{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\hat{e}_r\frac{ds}{rdt} = -\hat{e}_r\frac{v_\theta}{r}$. Therefore acceleration in a circular path in terms of polar unit vectors is given as

$$\vec{a} = -\frac{v^2}{r}\hat{e}_r + \frac{dv_\theta}{dt}\hat{e}_\theta$$

A particle moving in a circular trajectory has two kinds of acceleration. The first term is acceleration due to change of direction and is called centripetal or radial acceleration $\left(\vec{a}_c = \vec{a}_r = -\frac{v^2}{r}\hat{e}_r\right)$. Since the direction of $\hat{e}_r$ is radially outward, the direction of centripetal acceleration is always towards the center (radially inward). The second term is acceleration due to change of speed which is known as tangential acceleration $\left(\vec{a}_t = \vec{a}_\theta = \frac{dv_\theta}{dt}\hat{e}_\theta\right)$. Its direction is counter clockwise if either the particle is moving counter clockwise direction and the speed is increasing or it is moving in a clockwise direction and the speed is decreasing. Its direction is clockwise if either it is moving in a counter clockwise direction and the speed is decreasing or it is moving in a clockwise direction and the speed is increasing. The magnitudes of centripetal and tangential acceleration are respectively given as

$$a_c = \frac{v^2}{r} \text{ and } a_t = \left|\frac{dv}{dt}\right|$$

Since the direction of centripetal acceleration is radial and the direction of tangential acceleration is tangential, centripetal and tangential acceleration are always perpendicular to each other. Therefore the magnitude of the acceleration can be obtained from Pythagorean theorem.

$$a = \sqrt{a_c^2 + a_t^2} = \sqrt{\left(\frac{v^2}{r}\right)^2 + \left(\frac{dv}{dt}\right)^2}$$

Applying Newton's second law $\left(\vec{F} = m\vec{a}\right)$, the force acting on a particle moving in a circular path may be given in terms of polar unit vectors as

$$\vec{F} = -\frac{mv^2}{r}\hat{e}_r + m\frac{dv_\theta}{dt}\hat{e}_\theta$$

The first term is a component of the force directed towards the center known as the centripetal force $\left(\vec{F}_c = \frac{mv^2}{r}\hat{e}_r\right)$. The second term is a component of the force tangent to the circular trajectory known as tangential force $\left(\vec{F}_t = m\frac{dv_\theta}{dt}\hat{e}_\theta\right)$. The magnitudes of centripetal and tangential acceleration are respectively given as $F_c = \frac{mv^2}{r}$ and $F_t = m\left|\frac{dv}{dt}\right|$.

For a uniform circular motion (circular motion with constant speed) the tangential acceleration, $\frac{dv_\theta}{dt}$, is zero and thus $\vec{a} = \vec{a}_c = -\frac{v^2}{r}\hat{e}_r$ and $\vec{F} = \vec{F}_c = \frac{mv^2}{r}\hat{e}_r$ And of course the magnitudes are respectively given as $a = a_c = \frac{v^2}{r}$ and $F = F_c = \frac{mv^2}{r}$.

Example: A particle is revolving in a circular path of radius 10 m with a uniform speed of 5 m/s in a counterclockwise. When the particle is passing a point where the direction of its position vector is 37° west of north.

a) Express the position vector of the particle in terms of polar and Cartesian unit vectors.
Solution:
$r = 10$ m; $\theta = 90° + 37° = 127°$; $\vec{r} = ?$
Polar:

$$\vec{r} = r\hat{e}_r = 10\vec{e}_r \text{ m}$$

Cartesian:

$$\hat{e}_r = \cos(\theta)\hat{i} + \sin(\theta)\hat{j} = \cos(127°)\hat{i} + \sin(127°)\hat{j} = -0.6\hat{i} + 0.8\hat{j}$$

$$\vec{r} = 10\vec{e}_r \text{ m} = 10\left(-0.6\hat{i} + 0.8\hat{j}\right) \text{ m} = \left(-6\hat{i} + 8\hat{j}\right) \text{ m}$$

b) Express its velocity in terms of polar and Cartesian coordinates.
Solution:
$v = 5$ m/s; $\vec{v} = ?$
Polar:

$$\vec{v} = v_\theta \hat{e}_\theta = 5\hat{e}_\theta \text{ m/s}$$

Cartesian:

$$\hat{e}_\theta = -\sin(\theta)\hat{i} + \cos(\theta)\hat{j} = -\sin(127°)\hat{i} + \cos(127°)\hat{j} = -0.8\hat{i} - 0.6\hat{j}$$

$$\vec{v} = v_\theta \hat{e}_\theta = 5\left(-0.8\hat{i} - 0.6\hat{j}\right) \text{ m/s} = \left(-4\hat{i} - 3\hat{j}\right) \text{ m/s}$$

c) express its acceleration in terms of polar and Cartesian unit vectors.
Solution: Tangential acceleration is zero because it is moving with a constant speed.
$\frac{dv}{dt} = 0$; $\vec{a} = ?$
Polar:

$$\vec{a} = -\frac{v^2}{r}\hat{e}_r + \frac{dv_\theta}{dt}\hat{e}_\theta = \frac{5^2}{10}\hat{e}_r \text{ m/s}^2 = 2.5\hat{e}_r \text{ m/s}^2$$

Cartesian:

$$\vec{a} = 2.5\hat{e}_r \text{ m/s}^2 = 2.5\left(-0.6\hat{i} + 0.8\hat{j}\right) \text{ m/s}^2 = \left(-1.5\hat{i} + 2\hat{j}\right) \text{ m/s}^2$$

Example: A particle of mass 3 kg travelling in a circular path of radius 4m increased its speed uniformly from 2 m/s to 12 m/s in a counterclockwise direction in 5 seconds. By the time the direction of its position vector is north west, its speed is 10 m/s. Find its acceleration at this point in polar and Cartesian coordinates.

Solution: Since the tangential acceleration is uniform $\frac{dv_\theta}{dt} = \frac{\Delta v_\theta}{\Delta t} = \frac{v_{\theta f} - v_{\theta i}}{t}$.

$m = 3$ kg; $r = 4$ m; $v_{\theta i} = 2$ m/s; $v_{\theta f} = 12$ m/s; $\theta = 135°$; $v = 10$ m/s; $t = 5$ s; $\vec{a} = ?$

Polar:

$$\frac{dv_\theta}{dt} = \frac{\Delta v_\theta}{\Delta t} = \frac{v_{\theta f} - v_{\theta i}}{t} = \frac{12-2}{5} \text{ m/s}^2 = 2 \text{ m/s}^2$$

$$\vec{a} = -\frac{v^2}{r}\hat{e}_r + \frac{dv_\theta}{dt}\hat{e}_\theta = \left(-\frac{10^2}{4}\hat{e}_r + 2\hat{e}_\theta\right) \text{ m/s}^2 = \left(-25\hat{e}_r + 2\hat{e}_\theta\right) \text{ m/s}^2$$

Cartesian:

$$\hat{e}_r = \cos(\theta)\hat{i} + \sin(\theta)\hat{j} = \cos(135)\hat{i} + \sin(135)\hat{j} = -0.7\hat{i} + 0.7\hat{j}$$

$$\hat{e}_\theta = -\sin(\theta)\hat{i} + \cos(\theta)\hat{j} = -\sin(135)\hat{i} + \cos(135)\hat{j} = -0.7\hat{i} - 0.7\hat{j}$$

$$\vec{a} = \left(-25\hat{e}_r + 2\hat{e}_\theta\right) \text{ m/s}^2 = \left[-25\left(-0.7\hat{i} + 0.7\hat{j}\right) + 2\left(-0.7\hat{i} - 0.7\hat{j}\right)\right] \text{ m/s}^2 = \left(16\hat{i} - 18.9\hat{j}\right) \text{ m/s}^2$$

Practice Quiz 6.1

Choose the best answer

1. The direction of the polar *r*-unit vector for a point located on the positive x-axis is
   A) north
   B) None of the other choices are correct.
   C) east
   D) west
   E) south

2. The direction of the rate of change of the polar *r*-unit vector with respect to θ at a point on the negative y-axis is.
   A) east
   B) north
   C) west
   D) south
   E) None of the other choices are correct.

3. The distance between a certain point and the origin of an xy-coordinate plane is *20.5* m. The direction of the line joining the point and the origin is *60*° south of west. Express the position vector of the particle in terms of Cartesian unit vectors.
   A) *-18.992* m ***i*** + *-17.754* m ***j***
   B) *-10.25* m ***i*** + *-24.305* m ***j***
   C) *-10.25* m ***i*** + *-17.754* m ***j***
   D) *-16.977* m ***i*** + *-17.754* m ***j***
   E) *-10.25* m ***i*** + *-19.795* m ***j***

4. Express the position vector of the point whose Cartesian coordinates are $(3.1,\ 45)$ in terms of polar unit vectors.
   A) $3.1\ \boldsymbol{e}_r$
   B) $45°\ \boldsymbol{e}_\theta$
   C) $45.107\ \boldsymbol{e}_r + 86.059°\ \boldsymbol{e}_\theta$
   D) $86.059°\ \boldsymbol{e}_\theta$
   E) $45.107\ \boldsymbol{e}_r$

5. An object is revolving in a circular path of radius 4.5 m centered at the origin with a uniform speed of 14.5 m / s in a clockwise direction. By the time its position vector makes an angle of $60°$ with the positive x-axis, express its velocity in terms of polar unit vectors.
   A) -14.5 m / s $\boldsymbol{e}_r$
   B) 14.5 m / s $\boldsymbol{e}_\theta$
   C) -14.5 m / s $\boldsymbol{e}_\theta$
   D) 4.5 m $\boldsymbol{e}_r$ - 14.5 m / s $\boldsymbol{e}_\theta$
   E) -14.5 m / s $\boldsymbol{e}_r$

6. An object is revolving in a circular path of radius *4.5* m centered at the origin with a uniform speed of *15.8* m / s in a counterclockwise direction. By the time the direction of the position vector *55*° south of east, express its velocity in terms of Cartesian unit vectors.
   A) (*21.157* $\boldsymbol{i}$ + *9.063* $\boldsymbol{j}$) m / s
   B) (*12.943* $\boldsymbol{i}$ + *9.063* $\boldsymbol{j}$) m / s
   C) (*14.905* $\boldsymbol{i}$ + *9.063* $\boldsymbol{j}$) m / s
   D) (*12.943* $\boldsymbol{i}$ + *8.004* $\boldsymbol{j}$) m / s
   E) (*12.943* $\boldsymbol{i}$ + *12.109* $\boldsymbol{j}$) m / s

7. An object is revolving in a circular path of radius *6.3* m centered at the origin with a uniform speed of *17.3* m / s in a counterclockwise direction. By the time its position vector makes an angle of *45*° with the positive x-axis, express its acceleration in terms of polar unit vectors.
   A) *47.506* m / s^2 $\boldsymbol{e}_r$
   B) *47.506* m / s^2 $\boldsymbol{e}_\theta$
   C) *33.592* m / s^2 $\boldsymbol{e}_r$ + *33.592* m / s^2 $\boldsymbol{e}_\theta$
   D) *-47.506* m / s^2 $\boldsymbol{e}_r$
   E) *-47.506* m / s^2 $\boldsymbol{e}_\theta$

8. An object is revolving in a circular path of radius *8.2* m centered at the origin with a uniform speed of *13.3* m / s in a counterclockwise direction. By the time the direction of the position vector *70* ° south of west, express its acceleration in terms of Cartesian unit vectors.
   A) (*7.378* $\boldsymbol{i}$ + *20.271* $\boldsymbol{j}$) m / s^2
   B) (*7.378* $\boldsymbol{i}$ + *8.413* $\boldsymbol{j}$) m / s^2
   C) (*7.378* $\boldsymbol{i}$ + *23.25* $\boldsymbol{j}$) m / s^2
   D) (*13.118* $\boldsymbol{i}$ + *23.25* $\boldsymbol{j}$) m / s^2
   E) (*13.118* $\boldsymbol{i}$ + *20.271* $\boldsymbol{j}$) m / s^2

9. A particle is revolving in a circular path of radius *1.2* m centered at the origin. Its speed increased uniformly from *4.5* m / s to *14.5* m / s in *4* seconds. Express its final net acceleration in terms of polar unit vectors.
   A) (*-175.208* $\boldsymbol{e}_r$ + *4.022* $\boldsymbol{e}_\theta$) m / s^2
   B) (*-195.564* $\boldsymbol{e}_r$ + *4.022* $\boldsymbol{e}_\theta$) m / s^2
   C) (*-74.868* $\boldsymbol{e}_r$ + *3.131* $\boldsymbol{e}_\theta$) m / s^2
   D) (*-175.208* $\boldsymbol{e}_r$ + *2.5* $\boldsymbol{e}_\theta$) m / s^2
   E) (*-195.564* $\boldsymbol{e}_r$ + *2.5* $\boldsymbol{e}_\theta$) m / s^2

10. A particle is revolving in a circular path of radius *1.2* m centered at the origin with a uniform acceleration. Its speed increases by *12.5* m / s every *7.3* seconds. By the time it passes point P on the circle its speed is *4.1* m / s. If the direction of the position vector of point P is *25* ° west of north, express the net acceleration at point P in terms of Cartesian unit vectors.
    A) (*4.368* ***i*** + *-10.978* ***j***) m / s^2
    B) (*4.368* ***i*** + *-13.42* ***j***) m / s^2
    C) (*4.368* ***i*** + *-4.464* ***j***) m / s^2
    D) (*5.284* ***i*** + *-10.978* ***j***) m / s^2
    E) (*5.284* ***i*** + *-13.42* ***j***) m / s^2

11. A particle is displaced from point A to point B via a circular path of radius *19.5* m centered at the origin. The position vectors of point A and B make *15*° and *120*° with the positive x-axis respectively. Obtain the displacement of the particle.
    A) (*-6.794* ***i*** + *20.278* ***j***) m
    B) (*-6.794* ***i*** + *11.841* ***j***) m
    C) (*-28.586* ***i*** + *20.278* ***j***) m
    D) (*-28.586* ***i*** + *11.841* ***j***) m
    E) (*-45.931* ***i*** + *17.58* ***j***) m

6.3 EXAMPLES OF APPLICATIONS OF NEWTON'S SECOND LAW TO CIRCULAR MOTION

Example: A pendulum of mass m length ℓ and inclination θ is revolving in horizontal circle as shown. Obtain an expression for its speed in terms of its length and inclination.

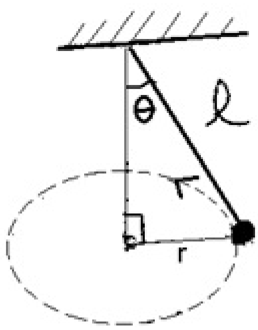

Figure 6.2

Solution: Let r be the radius of the horizontal circle. Then $r = \ell \sin(\theta)$. The forces acting on object are its weight $(\vec{w})$ and the tension in the string $(\vec{T})$ Let the forces be expresses in terms of the radially outward unit vector $(\hat{e}_r)$ and a unit vector perpendicular to the plane of motion $(\hat{j})$ That is $\vec{F} = F_r \hat{e}_r + F_y \hat{j}$. And if θ is angle measured with respect to $\hat{e}_r$, then $F_r = F\cos(\theta)$ and $F_y = F\sin(\theta)$ Since the speed is uniform, tangential acceleration is zero.

$$\theta_w = -90°;\ \theta_T = \theta + 90°;\ \frac{dv}{dt} = 0;\ v = ?$$

$$\vec{w} = -m|g|\hat{j} \Rightarrow F_r = 0 \text{ and } F_y = -m|g|$$

$$\vec{T} = T\cos(\theta + 90°)\hat{e}_r + T\sin(\theta + 90°)\hat{j} = -T\sin(\theta)\hat{e}_r + T\cos(\theta)\hat{j}$$

$$\Rightarrow T_r = -T\sin(\theta) \text{ and } T_y = T\cos(\theta)$$

$$\vec{a} = -\frac{v^2}{r}\hat{e}_r + \frac{dv}{dt}\hat{e}_\theta = -\frac{mv^2}{\ell\sin(\theta)}\hat{e}_r \Rightarrow a_r = -\frac{v^2}{\ell\sin(\theta)} \text{ and } a_y = 0$$

$$w_y + T_y = ma_y$$

$$-m|g| + T\cos(\theta) = 0$$

$$T\cos(\theta) = m|g|$$

$$w_r + T_r = ma_r$$

$$-T\sin(\theta) = -m\frac{v^2}{r}$$

$$T\sin(\theta) = m\frac{v^2}{r}$$

$$T\sin(\theta) = m\frac{v^2}{\ell\sin(\theta)} \text{ and } T\cos(\theta) = m|g| \Rightarrow v^2 = |g|\ell\sin(\theta)\tan(\theta)$$

$$v = \sqrt{\ell|g|\sin(\theta)\tan(\theta)}$$

Example: A car is revolving in a circular path of radius r on a horizontal surface. The coefficient of friction between the tires and the ground is μ. obtain an expression for the maximum uniform speed by which the car can make it without skidding.

Solution: The force responsible for the circular motion is friction which is directed towards the center. Forces acting are its weight $(\vec{w})$, normal force $(\vec{N})$ and friction $(\vec{f})$. Let the forces be expressed in terms of a radially outward unit vector $(\hat{e}_r)$ and a unit vector perpendicular to the plane of motion $(\hat{j})$. Since the speed is uniform tangential acceleration is zero.

$$\frac{dv}{dt} = 0;\ v_{\max}(r,\mu) = ?$$

$$\vec{w} = -m|g|\,\hat{j} \Rightarrow w_r = 0 \text{ and } w_y = 0$$

$$\vec{N} = N\hat{j} \Rightarrow N_r = 0 \text{ and } N_y = N$$

$$\vec{f} = -f\hat{e}_r = -\mu N\hat{e}_r \Rightarrow f_r = -\mu N \text{ and } f_y = 0$$

$$\vec{a} = -\frac{{v_{\max}}^2}{r}\hat{e}_r \Rightarrow a_r = -\frac{{v_{\max}}^2}{r} \text{ and } a_y = 0$$

$$w_y + N_y + f_y = ma_y$$

$$-m|g| + N = 0 \Rightarrow N = m|g|$$

$$w_r + N_r + f_r = ma_r$$

$$-\mu N = -m\frac{{v_{\max}}^2}{r} \Rightarrow v^2 = \frac{\mu N r}{m}$$

$${v_{\max}}^2 = \frac{\mu N r}{m} \text{ and } N = m|g| \Rightarrow v_{\max} = \sqrt{\mu|g|r}$$

Example: It is possible for a car to turn on a circular path on a horizontal plane in a frictionless road if the road is banked, because the component of the normal force directed towards the center of curvature will contribute to the centripetal force. Assuming no friction, what should the banking angle of the road, θ, be if the car is to turn with a uniform maximum speed v in a curve of radius of curvature r?

Solution: Let the mass of the car be denoted by m. The forces acting are its weight $(\vec{w})$ and normal force $(\vec{N})$. Let the forces be expressed in terms of a radially outward unit vector $(\hat{e}_r)$ and a unit vector perpendicular to the plane of motion $(\hat{j})$. The angle formed between $\hat{e}_r$ and the normal force is θ + 90°. Since the speed is uniform tangential acceleration is zero.

$$\theta_w = -90°;\ \theta_N = \theta + 90°;\ \frac{dv}{dt} = 0;\ \theta(r,v) = ?$$

$$\vec{w} = -m|g|\hat{j} \Rightarrow w_r = 0 \text{ and } w_y = -m|g|$$

$$\vec{N} = N\cos(\theta+90°)\hat{i} + N\sin(\theta+90°)\hat{j} = -N\sin(\theta)\hat{i} + N\cos(\theta)\hat{j}$$

$$\Rightarrow N_r = -N\sin(\theta) \text{ and } N_y = N\cos(\theta)$$

$$\vec{a} = -\frac{v^2}{r}\hat{e}_r \Rightarrow a_r = -\frac{v^2}{r} \text{ and } a_y = 0$$

$$w_y + N_y = ma_y$$

$$-m|g| + N\cos(\theta) = 0 \Rightarrow \cos(\theta) = \frac{m|g|}{N}$$

$$w_r + N_r = ma_r$$

$$-N\sin(\theta) = -m\frac{v^2}{r} \Rightarrow \sin(\theta) = \frac{mv^2}{rN}$$

$$\sin(\theta) = \frac{mv^2}{rN} \text{ and } \cos(\theta) = \frac{m|g|}{N} \Rightarrow \tan(\theta) = \left(\frac{v^2}{rg}\right)$$

$$\theta = \tan^{-1}\left(\frac{v^2}{r|g|}\right)$$

Example: A car is turning in a banked road with an angle of inclination θ and radius of curvature r. If the coefficient of friction between the road and the tires is μ, find an expression for the maximum speed by which it can make it without getting out of the circular trajectory.

Solution: The forces acting on the car are its weight $(\vec{w})$ friction $(\vec{f})$ and normal force. The direction of friction is parallel to the surface of the banked road. Let's express the forces in terms of a radially outward unit vector $(\hat{e}_r)$ and a unit vector perpendicular to the plane of motion. The angles formed by friction and normal force with respect to $\hat{e}_r$ are $\theta + 180°$ and $\theta + 90°$ respectively. Since the speed is uniform, tangential acceleration is zero.

$$\theta_w = -90^\circ;\ \theta_N = \theta + 90°;\ \theta_f = \theta + 180°;\ \frac{dv}{dt} = 0;\ v_{\max}(r, \theta, \mu) = ?$$

$$\vec{w} = -m|g|\hat{j} \Rightarrow w_r = 0 \text{ and } w_y = -m|g|$$

$$\vec{N} = N\cos(\theta + 90°)\hat{i} + N\sin(\theta + 90°)\hat{j} = -N\sin(\theta)\hat{i} + N\cos(\theta)\hat{j}$$

$$\Rightarrow N_r = -N\sin(\theta) \text{ and } N_y = N\cos(\theta)$$

$$\vec{f} = f\cos(\theta + 180°)\hat{e}_r + f\sin(\theta + 180°)\hat{j} = -\mu N\cos(\theta)\hat{e}_r - \mu N\sin(\theta)\hat{j}$$

$$\Rightarrow f_r = -\mu N\cos(\theta) \text{ and } f_y = -\mu N\sin(\theta)$$

$$\vec{a} = -\frac{v^2}{r}\hat{e}_r \Rightarrow a_r = -\frac{v^2}{r} \text{ and } a_y = 0$$

$$w_y + N_y + f_y = ma_y$$

$$-m|g| + N\cos(\theta) - \mu N\sin(\theta) = 0 \Rightarrow \cos(\theta) - \mu\sin(\theta) = \frac{m|g|}{N}$$

$$w_r + N_r + f_r = ma_r$$

$$-N\sin(\theta) - \mu N\cos(\theta) = -\frac{mv^2}{r} \Rightarrow \sin(\theta) + \mu\cos(\theta) = \frac{mv^2}{rN}$$

$$\sin(\theta) + \mu\cos(\theta) = \frac{mv^2}{rN} \text{ and } \cos(\theta) - \mu\sin(\theta) = \frac{m|g|}{N} \Rightarrow$$

$$v = \sqrt{r|g|\left(\frac{\sin(\theta) + \mu\cos(\theta)}{\cos(\theta) - \mu\sin(\theta)}\right)}$$

Example: An object of mass 3 kg is revolving in a vertical circle of radius 2 m by means of a rod with a uniform speed of 10 m/s. Calculate the tension in the rod when the rod makes an angle of 53° with the positive x-axis. Express the tension in terms of Cartesian unit vectors.

Solution: Forces acting on the object are its weight $(\vec{w})$ and the tension in the rod $(\vec{T})$. Unlike a string that can support tension along the string only, a rod can support tensions in other directions. Tangential acceleration is zero because it is revolving with a constant speed. $m = 3$ kg; $r = 2$ m; $v = 10$ m/s; $\theta_w = -90°$; $\theta = 53°$; $\vec{T} = ?$

$$\vec{w} = -m|g|\hat{j} = -3\times 9.8\hat{j} \text{ N}$$

$$\vec{a} = -\frac{v^2}{r}\hat{e}_r = -\frac{10^2}{2}\hat{e}_r \text{ m/s}^2 = -50\hat{e}_r \text{ m/s}^2$$

$$\vec{T} + \vec{w} = m\vec{a}$$

$$\vec{T} - 29.4\hat{j} \text{ N} = (3 \text{ kg})\left(-50\vec{e}_r \text{ m/s}^2\right) = -150\vec{e}_r \text{ N}$$

$$\vec{T} = \left(29.4\hat{j} - 150\hat{e}_r\right) \text{ N}$$

$$\hat{e}_r = \cos(\theta)\hat{i} + \sin(\theta)\hat{j} = \cos(53°)\hat{i} + \sin(53°)\hat{j} = 0.6\hat{i} + 0.8\hat{j}$$

$$\vec{T} = \left(29.4\hat{j} - 150\left[0.6\hat{i} + 0.8\hat{j}\right]\right) \text{ N} = \left(-90\hat{i} - 90.6\hat{j}\right) \text{ N}$$

Example: An object of mass m is revolving in a vertical circle by means of radius r by means of a string.

a) By the time the position vector of the object makes an angle θ with the positive x-axis, find expression for its tangential acceleration.
Solution: The forces acting are its weight $(\vec{w})$, and the tension in the string $(\vec{T})$. Let the forces be expressed in terms of $\hat{e}_r$ and $\hat{e}_\theta$ A string can support only tensions along the string; That is, the tension in a string can only have a radial component. The weight forms an angle of $\theta+90°$ with the radially out unit vector $\hat{e}_r$ in a clockwise direction.
$T_\theta = 0$; $\theta_w = -(\theta+90°)$; $a_t = a_\theta(r,\theta) = ?$

$$\vec{T} = -T\hat{e}_r \Rightarrow T_r = -T \text{ and } T_\theta = 0$$

$$\vec{w} = m|g|\left[\cos\left(-(\theta+90°)\right)\hat{e}_r + \sin\left(-(\theta+90°)\right)\hat{e}_\theta\right] = m|g|\left(-\sin(\theta)\hat{e}_r - \cos(\theta)\hat{e}_\theta\right)$$

$$\Rightarrow w_r = -mg\sin(\theta) \text{ and } w_\theta = -mg\cos(\theta)$$

$$T_\theta + w_\theta = ma_\theta$$

$$-m|g|\cos(\theta) = ma_\theta$$

$$a_\theta = -|g|\cos(\theta)$$

$$\vec{a}_\theta = -|g|\cos(\theta)\hat{e}_\theta$$

b) Obtain an expression for the minimum speed by which the object can make it to the top of the vertical circle.
Solution: For the minimum speed, the tension in the string should be zero. Therefore, for the minimum speed, the only force acting on the object at the top of the circle is its weight $(\vec{w})$. At the top of the circle θ = 90°. Therefore $\hat{e}_r = \hat{j}$ and $a_\theta = 0$
$T = 0$; $v_{\min} = ?$

$$\vec{w} = -m|g|\hat{j} \Rightarrow w_y = -m|g|$$

$$\vec{a} = -\frac{v_{\min}^{\,2}}{r}\hat{j} \Rightarrow a_y = -\frac{v_{\min}^{\,2}}{r}$$

$$\vec{w} = m\vec{a} \Rightarrow m|g| = \frac{mv_{\min}^{\,2}}{r}$$

$$v_{\min} = \sqrt{r|g|}$$

Practice Quiz 6.2

Choose the best answer

1. An object of mass 0.71 kg is revolving in a circular path of radius 3.2 m centered at the origin with a uniform speed of 15.8 m / s. By the time the direction of the position vector is $25°$ south of west, the force acting on the object is
   A) $(12.211\ \boldsymbol{i} + 11.957\ \boldsymbol{j})$ N
   B) $(50.199\ \boldsymbol{i} + 23.408\ \boldsymbol{j})$ N
   C) $(50.199\ \boldsymbol{i} + 19.755\ \boldsymbol{j})$ N
   D) $(90.041\ \boldsymbol{i} + 19.755\ \boldsymbol{j})$ N
   E) $(90.041\ \boldsymbol{i} + 23.408\ \boldsymbol{j})$ N

2. A particle of mass 0.93 kg is revolving in a circular path of radius 3.2 m centered at the origin. Its speed increased uniformly from 8.2 m / s to 15.8 m / s in 3 seconds. The initial force acting on the object is
   A) $(-19.542\ \boldsymbol{e}_r + 3.026\ \boldsymbol{e}_\theta)$ N
   B) $(-11.631\ \boldsymbol{e}_r + 3.833\ \boldsymbol{e}_\theta)$ N
   C) $(-22.679\ \boldsymbol{e}_r + 3.026\ \boldsymbol{e}_\theta)$ N
   D) $(-22.679\ \boldsymbol{e}_r + 2.356\ \boldsymbol{e}_\theta)$ N
   E) $(-19.542\ \boldsymbol{e}_r + 2.356\ \boldsymbol{e}_\theta)$ N

3. A particle of mass *0.58* kg is revolving in a circular path of radius *7.3* m centered at the origin with a uniform acceleration. Its speed increases by *17.3* m / s every *3.2* seconds. By the time it passes point P on the circle its speed is *9.6* m / s. If the direction of the position vector of point P is *55°* west of north, the force acting on the object at point P is
   A) $(1.646\ \boldsymbol{i} + -6.768\ \boldsymbol{j})$ N
   B) $(4.2\ \boldsymbol{i} + -6.768\ \boldsymbol{j})$ N
   C) $(1.646\ \boldsymbol{i} + -1.826\ \boldsymbol{j})$ N
   D) $(5.409\ \boldsymbol{i} + -5.341\ \boldsymbol{j})$ N
   E) $(4.2\ \boldsymbol{i} + -1.826\ \boldsymbol{j})$ N

4. A car of mass *2600* kg is turning on a curve of radius of curvature *100* m. The coefficient of friction between the tires of the car and the ground is *0.15*. Calculate the maximum speed by which the car can make it without sliding.
   A) *12.124* m / s
   B) *10.841* m / s
   C) *19.484* m / s
   D) *22.563* m / s
   E) *16.682* m / s

5. An object of mass *0.2* kg is revolving in a circular path of radius *0.5* m on a friction less table by means of a string which is attached to a hanging object of mass *4.5* kg (the string is attached to the hanging object through a hole on the table at the center of the circular path). Calculate the speed with which the object on the table is revolving on the circular path.
   A) *8.122* m / s
   B) *10.5* m / s
   C) *4.645* m / s
   D) *16.105* m / s
   E) *13.18* m / s

6. A *0.8* kg pendulum of length *0.2* m that makes an angle of *20* deg with the vertical is revolving in a horizontal circle. Calculate the speed with which it is revolving on the horizontal circle.
   A) *0.634* m / s
   B) *0.335* m / s
   C) *0.494* m / s
   D) *0.881* m / s
   E) *0.791* m / s

7. An object of mass *4* kg is revolving in a vertical circle of radius *1.5* m. Calculate the minimum speed at the top of the circle by which the object can make it without the string slacking.
   A) *6.291* m/s
   B) *2.489* m/s
   C) *3.834* m/s
   D) *0.936* m/s
   E) *3.011* m/s

8. What should the banking angle of a friction-less curved road of radius of curvature *50* m be, if a car is to make it without going off the road for speeds up to *15* m/s.
   A) *37.029°*
   B) *16.311°*
   C) *24.664°*
   D) *27.355°*
   E) *43.906°*

9. A car is turning on a banked curve of banking angle *20°*. The radius of curvature of the curve is *20* m. The coefficient of friction between the road and the tires is *0.225*. Calculate the maximum speed by which the car can make it on the curve without sliding.
   A) *11.213* m/s
   B) *9.638* m/s
   C) *5.601* m/s
   D) *4.157* m/s
   E) *19.539* m/s

10. An object of mass *0.18* kg is being revolved in a vertical circle of radius *4.1* m with a uniform speed of *18.1* m/s by means of a rigid metal rod. Determine the tension in the rod, by the time the object is at a point whose position vector is directed *70°* south of east.
    A) $(-6.09\,\boldsymbol{i} + 3.425\,\boldsymbol{j})$ N
    B) $(-4.919\,\boldsymbol{i} + 15.279\,\boldsymbol{j})$ N
    C) $(-7.632\,\boldsymbol{i} + 10.2\,\boldsymbol{j})$ N
    D) $(-7.632\,\boldsymbol{i} + 15.279\,\boldsymbol{j})$ N
    E) $(-4.919\,\boldsymbol{i} + 10.2\,\boldsymbol{j})$ N

7 WORK AND ENERGY

Your goal for this chapter is to learn about the relationships between work and energy. $1.739\,\mathrm{Nm}$

Work (W) is a process of transfer of energy. The process of transfer of energy involves a force displacing an object. When a force displaces an object it is only the component of the force in the direction of the displacement that contributes to the transfer of energy. If an object is displaced by a small displacement $d\vec{r}$ under the influence of a force $\vec{F}$, then the work done (dW) by the force on the object is defined to be equal to $F_{\parallel}dr$ where $F_{\parallel}$ is the component of the force in the direction of the displacement and dr is the magnitude of the displacement. If the angle formed between the force and the displacement is θ, then $F_{\parallel} = F\cos(\theta)$ and the expression for the work may be written as $dW = Fdr\cos(\theta)$ which is equal to the dot product between $\vec{F}$ and $d\vec{r}$

$$dW = \vec{F} \cdot d\vec{r}$$

The work done over a finite displacement is obtained by integrating over the path of the displacement. In other words work is equal to the line integral of the force evaluated on the path of the object. If a particle is displaced from a point whose position vector is $\vec{r}_i$ to a point whose position vector is $\vec{r}_f$ under the influence of a force $\vec{F}$, then the work done (W) by the force on the object is given by

$$W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

If the force is constant, then it can be taken out of the integral and $W = \vec{F} \cdot \int_{\vec{r}_i}^{\vec{r}_f} d\vec{r}$. But $\int_{\vec{r}_i}^{\vec{r}_f} d\vec{r} = \vec{r}_f - \vec{r}_i = \Delta\vec{r}$. Therefore for the work done by a constant force the integral simplifies to

$$W = \vec{F} \cdot \Delta\vec{r}$$

The displacement $\Delta\vec{r}$ is the vector whose tail is at the initial location and whose head is at the final location. If the angle formed between the force and the displacement is θ, this can also be written as $W = F\Delta r\cos(\theta)$ where F and Δr are the magnitudes of the force and displacement respectively. Δr is the distance between the initial and final points irrespective of the path taken by the particle while being displaced from the initial to the final point. If the distance between the initial and final point is represented by d $(d = \Delta r)$, then the work done by a constant force may also be written as

$$W = Fd\cos(\theta)$$

The SI unit of measurement of work is defined to be Joule, abbreviated as J. Work can be positive, negative or zero depending on the value of $\cos(\theta)$ Work is positive (negative) if the angle between force and the displacement is acute (obtuse). In other words, work is positive (negative) if its component in the direction of displacement is parallel (opposite to the displacement.

Example: A particle is displaced on a circular path of radius 5 m centered at the origin from the point $(5,0)$ m to the point $(0,5)$ m. One of the forces acting on the particle during this displacement is 50 N 37° north of east. Calculate the work done on the particle by this force.

Solution:

$\vec{r}_i = (5,0)\ \text{m} = 5\hat{i}\ \text{m};\ \vec{r}_i = (0,5)\ \text{m} = 5\hat{j}\ \text{m};\ F = 10\ \text{N};\ \theta_F = 37°;\ W = ?$

$$\vec{F} = F\cos(\theta_F)\hat{i} + F\sin(\theta_F)\hat{j} = \left(10\cos(37°)\hat{i} + 10\sin(37°)\right)\ \text{N} = \left(8\hat{i} + 6\hat{j}\right)\ \text{N}$$

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = \left(5\hat{j} - 5\hat{i}\right)\ \text{m}$$

$$W = \vec{F}\cdot\Delta\vec{r} = \left(8\hat{i} + 6\hat{j}\right)\cdot\left(5\hat{j} - 5\hat{i}\right)\ \text{J} = (-40 + 30)\ \text{J} = -10\ \text{J}$$

Example: In each of the following, calculate the work done.

a) An object pulled to the right a distance of 5 m by a 20 N force that makes an angle of 60° with the horizontal right.
Solution:
$d = 5\ \text{m};\ F = 20\ \text{N};\ \theta = 60°;\ W = ?$

$$W = Fd\cos(\theta) = 20\times 5\cos(60°)\ \text{J} = 50\ \text{J}$$

b) An object displaced to the right by 5 m while being pulled backwards by a 20 N force that makes an angle of 60° with the horizontal left.
Solution:
$d = 5\ \text{m};\ F = 20\ \text{N};\ \theta = (180 - 60)° = 120°;\ W = ?$

$$W = Fd\cos(\theta) = 20\times 5\cos(120°)\ \text{J} = -50\ \text{J}$$

Example: An object of mass 2 kg is sliding down a 5 m inclined plane that makes an angle of 30° with the horizontal. The coefficient of friction between the object and the surface of the plane is 0.2.

a) Calculate the work done by gravity.
Solution: The angle formed between weight force and the displacement (θ_w) is $90° - 30° = 60°$
$m = 2\ \text{kg};\ d = 5\ \text{m};\ \theta = 30°;\ W_w = ?$

$$F_w = m|g| = 2\times 9.8\ \text{N} = 19.6\ \text{N}$$

$$W_w = Fdc\cos(\theta_w) = 19.6\times 5\cos(60°)\ \text{J} = 49\ \text{J}$$

b) Calculate the work done by the normal force.

Solution: The work done by the normal force is zero because the normal force is perpendicular to the displacement $(\cos(90°)=0)$.

c) Calculate the work done by the force of friction.

Solution: As shown in a previous example, the normal force of an object sliding in an inclined plane is equal to weight times the cosine of the inclination angle (θ). The direction of friction is opposite to that of the displacement.

$\mu = 0.2$; θ_f=180°; $W_f = ?$

$$F_f = f = \mu N = \mu m|g|\cos(\theta) = 0.2\times 2\times 9.8\cos(60°)\ \text{N} = 1.96\ \text{N}$$

$$W_f = F_f d\cos(\theta_f) = 1.96\times 5\cos(180°)\ \text{J} = -9.8\ \text{J}$$

7.1 WORK DONE BY A VARIABLE FORCE IN ONE DIMENSION

Let's consider a particle displaced in a straight line by a variable force. Using a coordinate system where the x-axis lies along the straight line displacement, $d\vec{r} = dx\hat{i}$. Therefore $dW = \vec{F}\cdot d\vec{r} = \vec{F}\cdot\left(dx\hat{i}\right) = F_x dx$ where F_x is the component of the force along the x-axis. Thus the work done in displacing the particle from an initial location x_i to a final location x_f is given by

$$W = \int_{x_i}^{x_f} F_x dx$$

This also implies work can be obtained from a graph of force versus displacement as the area enclosed between the force versus displacement curve and the displacement axis. Areas above (below) the displacement axis is taken to be positive (negative).

Example: A particle is displaced along the x-axis from the location x = 2 m to the location x = 5 under the influence of a variable force that varies with x according to the formula $\vec{F} = \left(2\text{ N/m}^2\right)x^2\hat{i} + \left(4\text{ N}\right)\hat{j}$. Calculate the work done.

Solution:

$x_i = 2$ m; $x_f = 5$ m; $F_x = \left(2\text{ N/m}^2\right)x^2$; $W = ?$

$$W = \int_{x_i}^{x_f} F_x dx = \int_{2\text{ m}}^{5\text{ m}} \left(2\text{ N/m}^2\right)x^2 dx = \left(\frac{2}{3}\text{ N/m}^2\right)x^3\Big|_{2\text{ m}}^{5\text{ m}} = \frac{2}{3}\left(5^3 - 2^3\right)\text{ J} = 78\text{ J}$$

7.2 WORK DONE BY A VARIABLE FORCE IN TWO DIMENSIONS

For a two dimensional displacement (displacement in a plane), $d\vec{r} = dx\hat{i} + dy\hat{j}$ and $dW = \vec{F}\cdot d\vec{r} = \left(F_x\hat{i} + F_y\hat{j}\right)\cdot\left(dx\hat{i} + dy\hat{j}\right) = F_x dx + F_y dy$ Therefore the work done by a variable force in displacement a particle from the location $\vec{r}_i = \left(x_i, y_i\right)$ to the location $\vec{r}_f = \left(x_f, y_f\right)$ is given as

$$W = \int_{(x_i,y_i)}^{(x_f,y_f)} \left(F_x dx + F_y dy\right)$$

This integral can be converted into an integral with a single variable using the relationship between y and x on the path of the displacement.

Example: A particle is displaced by the force $\vec{F} = a\left(xy\hat{i} + y^2\hat{j}\right)$ from $x = c_1$ to $x = c_2$ along the $y = b$ line. Calculate the work done.

Solution: On the path of the displacement $dy = 0$ because y is constant $\left(y = b\right)$. Therefore $F_x dx + F_y dy = F_x dx = \left(axydx\right) = abxdx$ and

$$W = \int_{c_1}^{c_2} abx\,dx = ab\frac{x^2}{2}\Big|_{c_1}^{c_2} = ab\left(\frac{c_1^{\,2}}{2} - \frac{c_2^{\,2}}{2}\right)$$

Example: A particle is displaced by the force $\vec{F} = ax^3y^2\hat{i} + by^3x\hat{j}$ along the path $y = 2x$ from $x = c_1$ to $x = c_2$. Calculate the work done.

Solution: $y = 2x \Rightarrow dy = 2dx$ on the path. Therefore

$F_x dx + F_y dy = ax^3y^2dx + by^3xdy = ax^3(2x)^2 dx + b(2x)^3 x(2dx) = (4ax^5 + 16bx^4)dx$ and

$$W = \int_{c_1}^{c_2} (4ax^5 + 16bx^4)dx = \frac{4a}{6}(c_2{}^6 - c_1{}^6) + \frac{16b}{5}(c_2{}^5 - c_1{}^5)$$

Practice Quiz 7.1

Choose the best answer

1. The SI unit of measurement for work is
   A) Volt
   B) Newton
   C) Joule
   D) Coulomb
   E) Pascal

2. An object is displaced by a distance of *15* m horizontally by a *800* N force that makes an angle of *70* deg with the horizontal. Calculate the work done by the force.
   A) *4104.242* J
   B) *1405.06* J
   C) *2231.75* J
   D) *6261.262* J
   E) *0* J

3. An object is displaced from the point (*5.2, 1.3*) m to the point (*19.2, 16.7*) m under the influence of the force (*39.2* ***i*** + *21.3* ***j***) N. Calculate the work done by the force.
   A) *782.545* J
   B) *1316.815* J
   C) *876.82* J
   D) *156.298* J
   E) *603.935* J

4. One of the forces acting on an object revolving in a circular path of radius *7.4* m is (*1.3* $\boldsymbol{i}$ + *15.2* $\boldsymbol{j}$) N. Calculate the work done by this force as the object goes from point A to point B if the directions of the position vectors of points A and B are *30* ° and *110* ° respectively.
   A) *37.835* J
   B) *45.272* J
   C) *12.052* J
   D) *67.431* J
   E) *-4.055* J

5. A bullet of mass *0.33* kg is fired from the ground making an angle of *25*° with the ground with a speed of *250* m / s. Calculate the work done on the object by gravitational force by the time it returns to the ground.
   A) *0* J
   B) *4108.648* J
   C) *2152.868* J
   D) *3683.753* J
   E) *2887.466* J

6. An object of mass *2* kg is sliding in an inclined plane of length *7.3* m and inclination *55*°. The coefficient of kinetic friction between the surfaces of the object and the plane is *0.1*. Calculate the work done by friction as the object slides from top to bottom.
   A) *-4.432* J
   B) *-7.015* J
   C) *-1.597* J
   D) *-14.641* J
   E) *-8.207* J

7. The force acting on a particle moving on the x-axis varies with distance according to the equation $\boldsymbol{F} = 7/x^3\ \boldsymbol{i}$ Calculate the work done on the particle as the particle is displaced from $x = 5.7$ m to $x = 15.7$ m.
   A) *0.082* J
   B) *0.143* J
   C) *0.055* J
   D) *0.121* J
   E) *0.094* J

8. The following is a graph of the force exerted on a particle versus the distance covered by the particle.

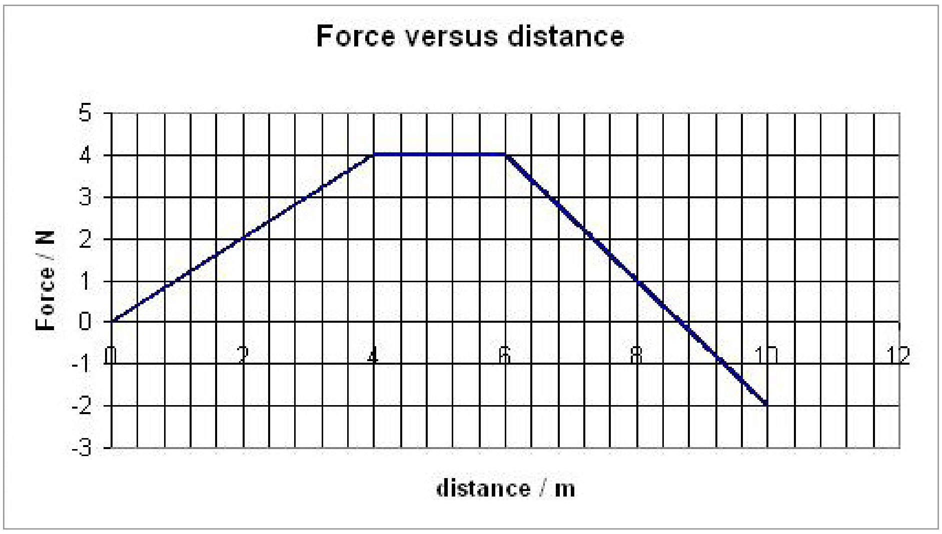

Figure 7.1

Calculate the work done on the particle as it is displaced from $x = 6$ m to $x = 10$ m.

A) *4* J
B) *8.333* J
C) *5* J
D) *6.667* J
E) *7.667* J

9. Calculate the work done by the force $\boldsymbol{F} = x^2y\ \boldsymbol{i} + x^2y^3\ \boldsymbol{j}$ while displacing a particle from $x = 6.1$ m to $x = 12.3$ m along the line $y = 8.3$ m.

A) *949.673* J
B) *2643.982* J
C) *7177.98* J
D) *4520.418* J
E) *5789.472* J

10. Calculate the work done by the force $\boldsymbol{F} = x^2y\ \boldsymbol{i} + xy^2\ \boldsymbol{j}$ while displacing a particle from $x = 5.7$ m to $x = 13.4$ m along the line $y = 5.7x$ m.

A) *1488306.509* J
B) *1983640.977* J
C) *239418.366* J
D) *807478.63* J
E) *1309469.781* J

7.3 WORK DONE BY THE FORCE DUE TO A SPRING

The dependence of the force due to a spring on displacement (extension or compression) is governed by Hook's Law. *Hook's law* states that the force due to a spring (F_s) is directly proportional and opposite to the displacement (x) of the spring.

$$F_s = -kx$$

k is a constant of proportionality called Hook's constant. It is constant for a given spring, but different for different springs. The unit of measurement for Hook's constant is N/m.

The work done by the force due to a spring (W_s) in extending (or compressing) the spring from $x = x_i$ to $x = x_f$ may be evaluated as $W_s = \int_{x_i}^{x_f} F_s dx = \int_{x_i}^{x_f} (-kx)\, dx$.

$$W_s = -\frac{k}{2}\left(x_f^2 - x_i^2\right)$$

Example: Calculate the work done by the force due to a spring of Hook's constant 100 N/m when the spring is compressed by 2 cm from its relaxed state.

Solution:
$x_i = 0;\ x_f = 0.02$ m; $k = 100$ N/m; $W_s = ?$

$$W_s = -\frac{k}{2}\left(x_f^2 - x_i^2\right) = -\frac{100}{2}\left(0.02^2 - 0\right)\ \mathrm{J} = -0.02\ \mathrm{J}$$

7.4 WORK-KINETIC ENERGY THEOREM

Net work done (W_{net}) on an object is defined to be the sum of all the work done by all the forces acting on the object or the work done by the net force acting on the object.

$$W_{net} = W_1 + W_2 + W_3 + \dots$$

Where W_i stands for the work done by the i^{th} force. If an object is displaced under the influence of a number of force $\vec{F}_1, \vec{F}_2, \vec{F}_3, \dots$, then the net work done on the objects is given as

$$W_{net} = \int \vec{F}_1 . d\vec{r} + \int \vec{F}_2 . d\vec{r} + \int \vec{F}_3 . d\vec{r} + \dots = \int \vec{F}_{net} . d\vec{r}$$

Where $\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots$. If all the forces remain constant during the displacement, then

$$W_{net} = \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots\right).\Delta\vec{r} = F_{net} \cdot \Delta\vec{r}$$

Example: A 10 kg object is displaced to the right on a horizontal surface by a distance of 5 m under the influence of the following forces: a 100 N force that makes an angle of 60° with the horizontal-right, a 20 N force pulling to the right and a 10 N force pulling to the left. If the coefficient of friction between the surfaces is 0.25, calculate the net work done on the object.

Solution: In addition to the forces stated, the object is being acted upon by its weight, normal force and friction. The work done by its weight and the normal force is zero because both forces are perpendicular to the displacement.
$m = 10$ kg; $d = 5$ m; $F_1 = 100$ N; $\theta_1 = 60°$; $F_2 = 20$ N; $\theta_2 = 0°$; $F_3 = 10$ N; $\theta_3 = 180°$; $\theta_f = 180°$; $\mu = 0.25$; $W_{net} = ?$

$$W_{net} = F_1 d\cos\theta_1 + F_2 d\cos\theta_2 + F_3 d\cos\theta_3 + fd\cos\theta_f$$

$$f = \mu N = \mu m|g| = 0.25 \times 10 \times 9.8 \text{ N} = 24.5 \text{ N}$$

$$W_{net} = 100 \times 5 \times \cos 60° + 20 \times 5 \times \cos 0° + 10 \times 5 \cos 180° + 24.5 \times 5 \cos 180° \text{ J} = 177.5 \text{ J}$$

Example: A particle is displaced from the point (2,3) m to the point (5,7) m under the influence of the following forces: $\vec{F}_1 = \left(2\hat{i} - 3\hat{j}\right)$ N, $\vec{F}_2 = \left(4\hat{i} + 6\hat{j}\right)$ N and $\vec{F}_3 = 6\hat{i}$ N. Calculate the net work done on the particle.

Solution:
$\vec{r}_i = (2,3)$ m $= \left(2\hat{i} + 3\hat{j}\right)$ m; $\vec{r}_f = (5,7)$ m $= \left(5\hat{i} + 7\hat{j}\right)$ m; $W_{net} = ?$

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = \left(5\hat{i} + 7\hat{j}\right)\text{ m} - \left(2\hat{i} + 3\hat{j}\right)\text{ m} = \left(3\hat{i} + 4\hat{j}\right)\text{ m}$$

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \left(2\hat{i} - 3\hat{j}\right)\text{ N} + \left(4\hat{i} + 6\hat{j}\right)\text{ N} + 6\hat{i}\text{ N} = \left(12\hat{i} + 3\hat{j}\right)\text{ N}$$

$$W_{net} = \vec{F}_{net} \cdot \Delta\vec{r} = \left(12\hat{i} + 3\hat{j}\right) \cdot \left(3\hat{i} + 4\hat{j}\right)\text{ J} = 48\text{ J}$$

Work kinetic energy theorem is a theorem that relates the net work done on an object with the change in motion energy produced by the work. Using Newton's second law $\left(F_{net} = ma = m\dfrac{dv}{dt}\right)$, $w_{net} = \int_{x_i}^{x_f} F_{net}dx = \int_{x_i}^{x_f} ma\,dx = \int_{x_i}^{x_f} m\dfrac{dv}{dt}dx$. But $\dfrac{dv}{dt}dx = dv\dfrac{dx}{dt} = vdv$. Therefore $W_{net} = \int_{x_i}^{x_f} mvdv$ which gives the following expression for the network done:

$$w_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

The expression $\frac{1}{2}mv^2$ is defined to be the kinetic energy (KE) of an object of mass m with a speed v.

$$KE = \frac{1}{2}mv^2$$

With this definition of kinetic energy, the net work done on an object is equal to the difference between the final and initial kinetic energy of the object. The *work-kinetic energy theorem* states that the net work done on an object is equal to the change in kinetic energy of the object.

$$W_{net} = KE_f - KE_i = \Delta KE$$

Example: A 2 kg object is sliding down a10 m inclined plane that makes an angle of 30° with the horizontal. It its speed at the bottom is 5 m/s, calculate the coefficient of friction between the surfaces.

Solution: The forces acting are weight, friction and normal force. The work done by the normal force is zero because the normal force is perpendicular to the displacement. The net work done is equal to the sum of the work done by its weight (W_w) and the work done by the force of friction (W_f). The angle formed between the weight and the displacement is 90° minus the inclination angle (θ)

$m = 2$ kg; $d = 10$ m; $\theta = 30°$; $v_i = 0$ (sliding); $v_f = 5$ m/s; $\mu = ?$

$$W_w = m|g|d\cos(90-\theta) = 2\times 9.8\times 10\cos(60°)\ \text{J} = 98\ \text{J}$$

$$W_w = m|g|d\cos(90-\theta) = 2\times 9.8\times 10\cos(60°)\ \text{J} = 98\ \text{J}$$

$$W_f = fd\cos(\theta_f) = (10\ \text{m})f\cos(180°) = -(10\ \text{m})f$$

$$f = \mu N = \mu m|g|\cos(\theta) = 2\times 9.8\cos(30)\mu \text{ N} = 16.97\mu \text{ N}$$

$$W_{net} = 62.5 \text{ J} = W_w + W_f = (98 - 10\times 16.97\mu) \text{ J}$$

$$\mu = 0.21$$

Example: A particle of mass 2 kg is displaced from the point (1, 1) m to the point (3, 4) m under the influence of the forces $\vec{F}_1 = (4\hat{i} + 6\hat{j})$ N and $\vec{F}_2 = (2\hat{i} + 10\hat{j})$. If its velocity at (1, 1) m is $(6\hat{i} + 8\hat{j})$ m/s, calculate its final speed at (3, 4) m.

Solution:

$m = 2$ kg; $\vec{r}_i = (1,1)$ m $= (\hat{i} + \hat{j})$ m; $\vec{r}_f = (3,4)$ m $= (3\hat{i} + 4\hat{j})$ m; $\vec{v}_i = (6\hat{i} + 8\hat{j})$ m/s; $v_f = ?$

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 = (4\hat{i} + 6\hat{j}) \text{ N} + (2\hat{i} + 10\hat{j}) \text{ N} = (6\hat{i} + 16\hat{j}) \text{ N}$$

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i = (3\hat{i} + 4\hat{j}) \text{ m} - (\hat{i} + \hat{j}) \text{ m} = (2\hat{i} + 3\hat{j}) \text{ m}$$

$$W_{net} = \vec{F}_{net}\cdot\Delta\vec{r} = (6\hat{i} + 16\hat{j})\cdot(2\hat{i} + 3\hat{j}) \text{ J} = 60 \text{ J}$$

$$w_{net} = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

$$v_i = \sqrt{\vec{v}_i \cdot \vec{v}_i} = \sqrt{\left(6\hat{i} + 8\hat{j}\right)\cdot\left(6\hat{i} + 8\hat{j}\right)} \text{ m/s} = \sqrt{100} \text{ m/s} = 10 \text{ m/s}$$

$$60 \text{ J} = \frac{1}{2} \times (2 \text{ kg}) v_f{}^2 - \frac{1}{2} \times 2 \times 10^2 \text{ J}$$

$$v_f = 12.64 \text{ m/s}$$

7.5 POWER

The power of a force is defined to be the rate of doing work by the force. If a force does a work of dW in a small time interval time dt, then the instantaneous power (P) is given as

$$P = \frac{dW}{dt}$$

The SI unit of measurement for power is J/s which is defined to be Watt, abbreviated as W.

Since $dW = \vec{F} \cdot d\vec{r}$, instantaneous power may also be written as $P = \vec{F} \cdot \frac{d\vec{r}}{dt}$ But $\frac{d\vec{r}}{dt}$ is the instantaneous velocity $(\vec{v})$ of the object. Thus, instantaneous power at a given instant of time is equal to the dot product between the force and the instantaneous velocity.

$$P = \vec{F} \cdot \vec{v}$$

Average power $(\overline{P})$ of a force in a given time interval Δt is equal to the ratio of the work done by the force during the time interval to the time interval; that is $\overline{P} = \frac{1}{\Delta t}\int \frac{dW}{dt}dt.$

$$\overline{P} = \frac{W}{\Delta t}$$

If the force is constant during the time interval, then $W = \vec{F} \cdot \Delta\vec{r}$ and $\overline{P} = \vec{F} \cdot \frac{\Delta\vec{r}}{\Delta t}$. But $\frac{\Delta\vec{r}}{\Delta t}$ is equal to the average velocity $(\overline{\vec{v}})$ of the object during the time interval Δt.

$$\overline{P} = \vec{F} \cdot \overline{\vec{v}}$$

Example: An object of mass 0.2 kg is thrown horizontally from a 2 m tall table with a speed of 10 m/s.

a) Calculate the average rate of doing work on the object by gravitational force by the time the object hits the ground.
Solution:
$m = 0.2$ kg; $\Delta y = -2$ m; $v_i = 10$ m/s; $\theta_i = 0$; $\overline{P} = ?$

$$\vec{F} = -m|g|\hat{j}$$

$$\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j}$$

$$\overline{P} = \vec{F}\cdot\frac{\Delta\vec{r}}{\Delta t} = -m|g|\hat{j}.\frac{\Delta xi + \Delta y\hat{j}}{\Delta t} = -m|g|\frac{\Delta y}{\Delta t}$$

Δt = t is the time taken to hit the ground.

$$v_{iy} = v_i \sin(\theta_i) = 10\sin(0) = 0$$

$$\Delta y = v_{iy} + \frac{1}{2}gt^2$$

$$-2 = -4.9t^2$$

$$t = \Delta t = 0.64 \text{ s}$$

$$\overline{P} = -m|g|\frac{\Delta y}{\Delta t} = -0.2\times 9.8\times\frac{-2}{0.64} \text{ W} = 6.125 \text{ W}$$

b) Calculate the instantaneous rate of doing work on the object by gravitational force by the time the object hits the ground.
Solution:
$P = ?$

$$\vec{v} = \vec{v}_f = v_{fx}\hat{i} + v_{fy}\hat{j}$$

$$P = \vec{F}\cdot\vec{v} = -m|g|\hat{j}\cdot\left(v_{fx}\hat{i} + v_{fy}\hat{j}\right) = -m|g|v_{fy}$$

$$v_{fy}^{\ 2} = v_{iy}^{\ 2} + 2g\Delta y = 2\times(-9.8)(-2) \text{ m}^2/\text{s}^2 = 39.2 \text{ m}^2/\text{s}^2$$

$$v_{fy} = 6.26 \text{ m/s}$$

$$P = -m|g|v_{fy} = -0.2\times 9.8(-6.26) \text{ W} = 12.23 \text{ W}$$

Practice Quiz 7.2

Choose the best answer

1. A certain spring extends by *0.1* m when an object of mass *0.8* kg hangs from it. Calculate the Hook's constant of the spring.
   A) *78.4* N / m
   B) *59.674* N / m
   C) *89.217* N / m
   D) *24.396* N / m
   E) *133.001* N / m

2. A certain spring extends by *0.032* m when an object of mass *8.2* kg hangs from it. The spring is extended by *0.77* m and then let go. Calculate the work done by the spring in changing this extended state to a state where it is extended by *0.038* m.
   A) *1228.775* J
   B) *211.388* J
   C) *742.647* J
   D) *514.221* J
   E) *333.326* J

3. An object was displaced from the point (*7, 14*) m to the point (*19.3, 14.2*) m under the influence of the forces (*25.1, 26.3*) N and an unknown constant force. If the net work done on the object is *16.1* J, calculate the work done by the unknown force.
   A) *-463.159* J
   B) *-297.89* J
   C) *-379.819* J
   D) *-495.509* J
   E) *-347.781* J

4. A 2 kg object is displaced to the right by a distance of *20* m under the influence of the following forces: a *6* N force pulling to the left, a *56* N force that makes an angle of *60* deg with the horizontal-right. Calculate the net work done on the object.
   A) *289.28* J
   B) *440* J
   C) *195.686* J
   D) *639.993* J
   E) *552.363* J

5. Calculate the speed of an object of mass *12* kg if its kinetic energy is *1200* J.
   A) *5.514* m / s
   B) *11.36* m / s
   C) *8.104* m / s
   D) *14.142* m / s
   E) *25.536* m / s

6. Under the influence of some forces, the speed of a *24* kg object changed from *15* m / s to *90* m / s. Calculate the net work done by the forces.
   A) *94500* J
   B) *0* J
   C) *115244.214* J
   D) *69623.147* J
   E) *0* J

7. An object of mass *4.5* kg is displaced horizontally to the right by a distance of *6* m under the influence of the following forces: A *44* N force pulling to the right and a *30* N force pulling to the left. If it started from rest, calculate its final speed.
   A) *6.11* m / s
   B) *11.219* m / s
   C) *1.79* m / s
   D) *4.07* m / s
   E) *9.246* m / s

8. An *2* kg object was displaced from the point (*6,2*) m to the point (*18.2, 14.2*) m under the influence of the forces (*23.4* ***i*** + *28.5* ***j***) N and (*16.1* ***i*** + *17.4* ***j***) N. If its initial velocity is (*2* ***i*** + *3* ***j***) m / s, calculate the magnitude of its final velocity.
   A) *46.315* m / s
   B) *36.095* m / s
   C) *12.951* m / s
   D) *42.765* m / s
   E) *32.479* m / s

9. A car of mass *2453* kg initially moving with a speed of *15.1* m / s was stopped in a distance of *24.8* m. Calculate the work done by friction.
   A) *-2.797e5* J
   B) *-4.121e5* J
   C) *-5.019e5* J
   D) *-4.587e5* J
   E) *-2.233e5* J

10. An object of mass *0.025* kg released from a height of *14.5* m has a speed of *8* m / s just before it hits the ground. Calculate the work done by air resistance.
   A) *-4.444* J
   B) *-2.117* J
   C) *-3.967* J
   D) *-1.097* J
   E) *-2.753* J

11. The power of the engine of a certain car is rated as *675* horse power (1 horse power = *746* W). How much work can the engine do in *0.53* hour?
   A) *624.364* MJ
   B) *960.773* MJ
   C) *228.934* MJ
   D) *1074.071* MJ
   E) *1330.16* MJ

12. An object of mass *1.6* kg is released from a height of *35* m. Calculate the instantaneous power of gravity just before it hits the ground.
   A) *51.13* W
   B) *410.684* W
   C) *619.155* W
   D) *749.679* W
   E) *264.057* W

8 POTENTIAL ENERGY AND CONSERVATION OF MECHANICAL ENERGY

Your goal for this chapter is to learn about the properties of special forces called conservative forces.

8.1 CONSERVATIVE FORCE

Conservative forces are forces for which the work done is independent of the path followed. It depends only on the potential energies of the objects at the initial and final locations. Potential energy is energy that an object possesses just because of its location (orientation, deformation…). Work done by a conservative force (W_c) is equal to the negative of the change in the potential energy (U) of the object.

$$W_c = -\Delta \mathrm{U} = -(U_f - U_i)$$

Or

$$\Delta U = -\int_{x_i}^{x_f} F_c dx$$

$U_i\left(U_f\right)$ is potential energy of the object at the initial (final) location of the object. Work done on a closed path is zero, because the initial and final locations are the same $(U_i = U_f = 0 \Rightarrow \Delta U = 0)$.

The work done by a conservative force in displacing an object by an infinitely small displacement, *dx*, may be given as $dW_c = F_c dx = -dU$ which implies

$$F_c = -\frac{dU}{dx}$$

A conservative force acting on an object is equal to the negative rate of change of its potential energy with respect to position. The potential energy can be obtained as a function of position by integrating this equation.

$$U(x) = -\int F_c(x)dx + C$$

Where C is an arbitrary integration constant. To specify a unique value of the potential energy at any point, a reference point $\left(x = x_0\right)$ where the potential energy is arbitrarily set to zero is needed. (The reference point is usually chosen in such a way that the value of C is zero. This means the reference point $x = x_0$ is chosen in such a way that $\int F_c(x)dx\Big|_{x=x_0} = 0$.) With $U\left(x_0\right)$ set to zero, $\int_{U(x_0)}^{U(x)} dU = U(x) - U\left(x_0\right) = U\left(x\right)$ and a unique value for the potential energy at any point can be obtained from

$$U\left(x\right) = -\int_{x_0}^{x} F_c(x')dx'$$

Example: The potential energy of a certain object varies with position according to the equation $U\left(x\right) = c_1 x^2 + c_2 x + c_3$ where c_1, c_2 and c_3 are constants Find an expression for the conservative force associated with this potential energy as a function of position $\left(x\right)$.

Solution:

$$F_c(x) = -\frac{dU}{dx} = -\frac{d}{dx}\left(c_1 x^2 + c_2 x + c_3\right) = -2c_1 x - c_2$$

Example: The conservative force acting on an object varies with position (x) according to the equation $F(x) = -ax^3$ where $a = 2\ \mathrm{N/m^3}$ Assuming the potential energy at the origin is zero, find an expression for the dependence of its potential energy on position. Also obtain its potential energy at x = 2 m.

Solution:

$x_0 = 0;\ U(x) = ?;\ U\big|_{x=2\mathrm{m}} = ?$

$$U(x) = -\int_{x_0}^{x} F_c(x')dx' = -\int_{0}^{x}\left(-a(x')^3\right)dx' = a\left.\frac{(x')^4}{4}\right|_0^x = a\frac{x^4}{4}$$

$$U(x = 2\ \mathrm{m}) = 2\frac{2^4}{4}\ \mathrm{J} = 8\ \mathrm{J}$$

Example: The conservative force acting on an object varies with position according to the equation $F(x) = -\frac{a_1}{x^3} + \frac{a_2}{x^2}$. Find an expression for its potential energy as a function of x assume the potential energy at infinity to be zero.

Solution:

$x_0 = \infty;\ U(x) = ?$

$$U(x) = -\int_{x_0}^{x} F_c(x')dx' = -\int_{\infty}^{x}\left(-\frac{a_1}{(x')^3} + \frac{a_2}{(x')^2}\right)dx' = \left.\left(-\frac{a_1}{2(x')^2} + \frac{a_2}{(x')}\right)\right|_{\infty}^{x} = -\frac{a_1}{2x^2} + \frac{a_2}{x}$$

Example: The conservative force acting on an object varies with position according to the equation $F(x) = -a_1x^2 + a_2$ where $a_1 = 3\ \mathrm{N/m^2}$ and $a_2 = 5\ \mathrm{N}$. Find the change in its potential energy and the work done by the conservative force as it is displaced from x = 2 m to x = 4m.

Solution:

$x_i = 2\ \mathrm{m};\ x_f = 4\ \mathrm{m};\ \Delta U = ?;\ W_c = ?$

$$\Delta U = -\int_{x_i}^{x_f} F_c dx = -\int_{2\ \mathrm{m}}^{4\ \mathrm{m}}\left(-a_1x^2 + a_2\right)dx = \left.\left(\frac{a_1x^3}{3} - a_2x\right)\right|_{2\ \mathrm{m}}^{4\ \mathrm{m}} = 46\ \mathrm{J}$$

$$W_c = -\Delta U = -46\ \mathrm{J}$$

8.2 GRAVITATIONAL POTENTIAL ENERGY

Gravitational force $\left(\vec{F}_g\right)$ acting on an object of mass m (weight) is given by $\vec{F}_g = -m|g|\hat{j}$. Therefore the change in the potential energy $\left(dU_g\right)$ of an object under the influence gravitational force when displaced by a small displacement $d\vec{r} = dx\hat{i} + dy\hat{j}$ is given as $dU_g = -\vec{F}_g \cdot d\vec{r} = -\left(-m|g|\hat{j}\right).\left(dx\hat{i} + dy\hat{j}\right) = m|g|dy$. Setting the potential energy at $y = 0$ to be zero (that is $y_0 = 0$), the gravitational potential energy may be given as a function of y as $U_g\left(y\right) = -\int_{y_o}^{y} \vec{F} \cdot d\vec{r} = \int_0^y m|g|dy$.

$$U_g\left(y\right) = m|g|y$$

The value of gravitational energy at a point depends on the choice of reference point (origin). The work done by gravitational force $\left(W_g\right)$ in displacing an object from the location $\left(x_i, y_i\right)$ to the location $\left(x_f, y_y\right)$ can be obtained as $W_g = -\Delta U_g = -\left(U_{gf} - U_{gi}\right) = -\left(m|g|y_f - m|g|y_i\right)$.

$$W_g = m|g|\left(y_f - y_i\right)$$

a) Calculate the gravitational potential energy of an object of mass 10kg located at a point whose position vector is $\vec{r} = \left(2\hat{i} + 6\hat{j}\right)$ m.
Solution:
$m = 10$ kg; $y = 6$ m; $U_g = ?$

$$U_g = m|g|y = 10 \times 9.8 \times 6 \text{ J} = 588 \text{ J}$$

Example: Calculate the work done by gravity on a 2 kg object when it is displaced from the location $\left(-2\hat{i} + 4\hat{j}\right)$ m to the location $\left(8\hat{i} + 2\hat{j}\right)$ m.

Solution:
$m = 2$ kg; $y_i = 4$ m; $y_f = 2$ m; $W_g = ?$

$$W_g = -m|g|\left(y_f - y_i\right) = -2 \times 9.8\left(2 - 4\right) = 39.2 \text{ J}$$

Example: Obtain an expression for gravitational force starting from the expression from gravitational potential energy.

Solution:
$U_g(y) = m|g|y$; $\vec{F}_g = ?$

$$F_{gy} = -\frac{dU_g}{dy} = -\frac{d\left(m|g|y\right)}{dy} = -m|g|$$

$$\vec{F}_g = -m|g|\hat{j}$$

8.3 ELASTIC POTENTIAL ENERGY

From Hook's Law, the force due to a spring depends on its displacement (extension or compression) according to the equation $\vec{F}_s = -kx\hat{i}$. Setting the elastic potential energy $\left(U_{el}\right)$ of the spring at x = 0 or relaxed position (that is $x_0 = 0$, the elastic potential energy can be obtained as a function of x as $U_{el}\left(x\right) = -\int_{x_0}^{x} F_{sx}\left(x'\right)dx' = -\int_{0}^{x}\left(-kx'\right)dx'$.

$$U_{el}\left(x\right) = \frac{1}{2}kx^2$$

The work done by the force due to a spring $\left(W_{el}\right)$ is equal to the negative of change in potential energy: $W_{el} = -\Delta U_{el} = -\left(U_{elf} - U_{eli}\right) = -\left(\frac{1}{2}kx_f^{\,2} - \frac{1}{2}kx_i^{\,2}\right)$.

$$W_{el} = -\frac{1}{2}k\left(x_f^{\,2} - x_i^{\,2}\right)$$

Example: Calculate the elastic potential energy stored by a spring of Hook's Constant 200 N/m when compressed by 2 cm.

Solution:
$k = 200$ N/m; $x = 0.02$ m; $U_{el} = ?$

$$U_{el} = \frac{1}{2}kx^2 = \frac{1}{2} \times 200 \times 0.02^2 \text{ J} = 4 \times 10^{-2} \text{ J}$$

Example: Calculate the work done by the force due to a spring of Hook's constant 100 N/m when the spring is extended from x = 1 cm to x = 6 cm.

Solution:
$x_i = 0.01$ m; $x = 0.06$ m; $k = 100$ N/m; $W_{el} = ?$

$$W_{el} = -\frac{1}{2}k\left(x_f^{\,2} - x_i^{\,2}\right) = -\frac{1}{2} \times 100\left(0.06^2 - 0.01^2\right) \text{ J} = -1.75 \text{ J}$$

8.4 CONDITIONS OF EQUILIBRIUM

An object is said to be in equilibrium (i.e. either rest or moving in a straight line with a constant speed) if the force acting on it is zero. If a particle is being acted upon by a conservative force, then since $F_{cx} = -\frac{dU}{dx}$, a particle will be in equilibrium at the point $x = x_0$, if the condition

$$\left.\frac{dU}{dx}\right|_{x=x_0} = 0$$

is satisfied. A particle acted upon by a conservative force will be in equilibrium at a point if the derivative of the potential energy with respect to position at the point is zero. In other words, the particle will be in equilibrium at the turning points of its potential energy. There are two kinds of equilibrium: stable and unstable equilibrium.

Stable equilibrium is equilibrium where the particle tends to return to its equilibrium position when displaced by a small displacement. This happens when the graph potential energy versus position opens upwards at the turning point. Mathematically, this happens when the second derivative of the potential energy with respect to position is positive at the point of equilibrium (turning point). Thus a particle will be in stable equilibrium at $x = x_0$ if the conditions

$$\left.\frac{dU}{dx}\right|_{x=x_0} = 0 \text{ and } \left.\frac{d^2U}{dx^2}\right|_{x=x_0} > 0$$

are satisfied. An example of a stable equilibrium is a ball in a valley. When it is is displaced a little bit, it returns to the valley.

Unstable Equilibrium is equilibrium where the particle tends to go away from its equilibrium position when displaced by a small displacement. This happens when the graph of potential energy as a function of position opens downward at the turning point. Mathematically this happens when the second derivative at the equilibrium position (turning point) is negative. Therefore a particle will be in unstable equilibrium at $x = x_0$ if the conditions

$$\left.\frac{dU}{dx}\right|_{x=x_0} = 0 \text{ and } \left.\frac{d^2U}{dx^2}\right|_{x=x_0} < 0$$

Are satisfied. An example of Unstable equilibrium is a ball at the top of a hill. If it is displaced a little bit, it goes down the hill

Example: The potential energy of a certain particle varies with position according to the equation $U(x) = ax^2 - b$ where $a = 1\ \text{J/m}^2$ and $b = 4$ J.

a) Find the point(s) of equilibrium.
Solution:
$x_0 = ?$

$$\left.\frac{dU}{dx}\right|_{x=x_0} = \frac{d}{dx}\left(ax^2 - b\right)\Big|_{x=x_0} = 2ax\big|_{x=x_0} = 2ax_0 = 0$$

$$x_0 = 0$$

b) Determine if the equilibrium point is stable or unstable.
Solution: If the point is a point of stable (unstable) equilibrium, then the second derivative should be positive (negative).

$$\left.\frac{d^2U}{dx^2}\right|_{x=x_0} = \frac{d}{dx}\left(\frac{dU}{dx}\right)\Big|_{x=0} = \frac{d}{dx}\left(2ax\right)\Big|_{x=0} = 2a = 2 \text{ J/m}^2 > 0$$

Therefore, the equilibrium point is a stable equilibrium point.
Example: The potential energy of a certain particle varies with position according to the equation $U(x) = ax^3 - bx$ where $a = \frac{1}{3}$ J/m^3 and $b = 4$ J/m.

a) Determine the points of equilibrium.
Solution:
$x_0 = ?$

$$\left.\frac{dU}{dx}\right|_{x=x_0} = \frac{d}{dx}\left(ax^3 - bx\right)\Big|_{x=x_0} = \left(3ax^2 - b\right)\Big|_{x=x_0} = 3ax_0^{\ 2} - b = 0$$

$$x_0 = \pm\sqrt{\frac{b}{3a}} = \pm\sqrt{\frac{4}{3\times\frac{1}{3}}} \text{ m} = \pm 2 \text{ m}$$

The equilibrium points are located at x = 2 and x = –2m.

b) Determine if the equilibrium points are stable or unstable.
Solution:
$x_0 = 2$ m

$$\left.\frac{d^2U}{dx^2}\right|_{x=x_0} = \frac{d}{dx}\left(\frac{dU}{dx}\right)\Big|_{x=2\text{ m}} = \frac{d}{dx}\left(\left(3ax^2 - b\right)\right)\Big|_{x=2\text{ m}} = 6ax\big|_{x=2\text{ m}} = 4 \text{ J/m}^2 > 0$$

Therefore the equilibrium point at $x = 2$ m is a stable equilibrium point.

$x_0 = -2$ m

$$\left.\frac{d^2U}{dx^2}\right|_{x=x_0} = \frac{d}{dx}\left(\frac{dU}{dx}\right)\bigg|_{x=-2\text{ m}} = 6ax\big|_{x=-2\text{ m}} = -4\text{ J/m}^2 < 0$$

Therefore the equilibrium point at $x = -2$ is an unstable equilibrium point.

8.5 CENTRAL FORCES

Central forces are forces whose magnitude depend on the distance (r) between the source (origin) and the point, and whose direction is along the direction of the position vector $(\vec{r})$ of the point. Central forces can generally be written as

$$\vec{F} = f(r)\hat{e}_r = f(r)\frac{\vec{r}}{r}$$

Where $f(r)$ an arbitrary functions of *r*. Gravitational and electrical forces are examples of central forces. Central forces are conservative. Therefore the work done by a central force is equal to the negative of the potential energy $(U(r))$ associated with it. That is $dU(r) = -f(r)\hat{e}_r \cdot d\vec{r}$. The path element $d\vec{r}$ can be expressed in terms of polar coordinates as follows. When θ is constant only the length of the position vector can change and the component of $d\vec{r}$ when θ is constant, can be written as $dr\vec{e}_r$. If *r* is kept constant, $\vec{r}$ can only rotate in a circle of radius *r*. If the arc length along the circular path is *ds*, then the component of $d\vec{r}$ along $\vec{e}_\theta$ can be written as $ds\vec{e}_\theta$. Thus $d\vec{r}$ can be written in terms of polar coordinates as $d\vec{r} = dr\vec{e}_r + rd\theta\vec{e}_\theta$. Therefore $dU(r) = -f(r)\hat{e}_r \cdot d\vec{r} = -f(r)\hat{e}_r \cdot (dr\hat{e}_r + rd\theta\hat{e}_\theta) = -f(r)dr$.

If the potential energy is set to be zero at $r = r_0$, then the potential energy associated with a central force is given as

$$U(r) = -\int_{r_0}^{r} f(r')dr'$$

A central force is equal to the negative derivative of its potential energy with respect to *r*.

$$\vec{F} = -\frac{dU(r)}{dr}\hat{e}_r$$

The direction of a force that changes the magnitude of a position vector only must be along $\hat{e}_r$.

Example: A central force is given as $\vec{F}(r) = -\frac{a}{r^2}\hat{e}_r$ where $a = 2\ \mathrm{Nm}^2$ Assuming the potential energy at infinity to be zero, calculate the potential energy of a particle located at the point $(3,4)$ m.

Solution:

$r_0 = \infty;\ \vec{r} = (3,4)\ \mathrm{m};\ f(r) = -\frac{a}{r^2};\ U(r) = ?$

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2}\ \mathrm{m} = 5\ \mathrm{m}$$

$$U(r) = -\int_{r_0}^{r} f(r')dr' = -\int_{\infty}^{r}\left(\frac{-a}{(r')^2}\right)dr' = -\frac{a}{r}\bigg|_{\infty}^{r} = -\frac{a}{r}$$

$$U(r = 5\ \mathrm{m}) = -\frac{2}{5}\ \mathrm{J}$$

Example: The potential energy of a particle varies on the distance from the origin according to the equation $U(r) = -\frac{a}{r}$ where a = 5 Jm. Calculate the force exerted on a particle located at the point $(3,4)$ m.

Solution:
$\vec{r} = (3,4)$ m; $F(r) = ?$

$$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} \text{ m} = 5 \text{ m}$$

$$\vec{F}(r) = -\frac{dU}{dr}\vec{e}_r = -\frac{d}{dr}\left(-\frac{a}{r}\right)\vec{e}_r = -\frac{a}{r^2}\vec{e}_r$$

$$\vec{F}(r = 5 \text{ m}) = -\frac{5}{5^2}\hat{e}_r \text{ N} = -\frac{1}{5}\hat{e}_r \text{ N}$$

Practice Quiz 8.1

Choose the best answer

1. In which of the following situation is the change in motion not caused by a conservative force?
   A) Pieces of paper being attracted by an electrically charged comb.
   B) An iron sample being attracted by a permanent magnet.
   C) Earth revolving around the sun.
   D) An object being pushed by a compressed spring.
   E) A book sliding on a table eventually coming to rest.

2. An object is displaced *21* m by an *84* N conservative force that makes an angle of *40* deg with the displacement. Calculate the change in its potential energy.
   A) *-1351.302* J
   B) *1351.302* J
   C) *-605.743* J
   D) *-446.197* J
   E) *446.197* J

3. An object of mass *40.3* kg is located at the top of an *18.9* m tall building. Calculate the gravitational potential energy of the object with respect to a point *1* m above the top of the building.
   A) *-394.94* J
   B) *394.94* J
   C) *42.336* J
   D) *0* J
   E) *-42.336* J

4. An object of mass *9* kg is sliding down a friction less inclined plane of length *3* m that makes an angle of *10* deg with the horizontal. Calculate the work done by gravitational force as the object slides from the top of the inclined plane to the ground.
   A) *16.634* J
   B) *79.164* J
   C) *45.947* J
   D) *52.324* J
   E) *70.625* J

5. Calculate the Hook's constant of a spring that stores *0.375* J of energy when compressed by *0.075* m.
   A) *242.711* N / m
   B) *228.789* N / m
   C) *133.333* N / m
   D) *42.143* N / m
   E) *96.953* N / m

6. The force acting on a particle moving along the x-axis depends on x according to the equation $\boldsymbol{F} = (2.4x - 3.1)\ \boldsymbol{i}$ Assuming the potential energy at the origin is zero, calculate the potential energy of the particle at $x = 7.4$ m.
   A) *-54.873* J
   B) *-17.508* J
   C) *-42.772* J
   D) *-29.149* J
   E) *-64.089* J

7. The potential energy of a certain particle varies along the x-axis according to the equation $U = 4.712x - 3.1$. Calculate the force acting on the particle at $x = 7.4$ m.
   A) *-1.858* N $\boldsymbol{i}$
   B) *-5.944* N $\boldsymbol{i}$
   C) *-5.472* N $\boldsymbol{i}$
   D) *-4.712* N $\boldsymbol{i}$
   E) *-3.295* N $\boldsymbol{i}$

8. The potential energy of a certain particle varies along the x-axis according to the equation $U = 8.4/x^2 - 4.7/x$. The particle will be in equilibrium when x is equal to
   A) *5.753* m
   B) *3.574* m
   C) *3.026* m
   D) *1.815* m
   E) *6.56* m

9. The potential energy of a certain particle varies along the x-axis according to the equation $U = -9.2x^3 + 8.4x$. The particle will be in stable equilibrium when x is equal to
   A) *0.552* m
   B) *0* m
   C) *-0.221* m
   D) *-0.552* m
   E) Two of the other choices are correct answers

10. The potential energy of a certain particle varies along the x-axis according to the equation $U = 6.1x^3 - 2.4x$. The particle will be in unstable equilibrium when x is equal to
    A) -0.158 m
    B) 0.362 m
    C) Two of the other choices are correct answers
    D) -0.362 m
    E) 0.158 m

11. A certain central force varies with distance from the origin according to the equation $\boldsymbol{F} = 8.1/r^3\ \boldsymbol{e}_r$. Assuming the potential at infinity to be zero, calculate the potential energy of a particle located at the point ($1.55,\ 0.64$) m.
    A) 1.686 J
    B) 1.44 J
    C) 2.125 J
    D) 0.289 J
    E) 2.487 J

12. The potential energy of a certain particle varies with distance from the origin according to the equation $U = 200/r^2$ calculate the force acting on a particle located at the point ($1.43,\ 0.64$) m.
    A) ($23.823\ \boldsymbol{i} + 42.493\ \boldsymbol{j}$) N
    B) ($94.945\ \boldsymbol{i} + 29.199\ \boldsymbol{j}$) N
    C) ($23.823\ \boldsymbol{i} + 29.199\ \boldsymbol{j}$) N
    D) ($94.945\ \boldsymbol{i} + 42.493\ \boldsymbol{j}$) N
    E) ($170.064\ \boldsymbol{i} + 48.83\ \boldsymbol{j}$) N

8.6 CONSERVATION OF MECHANICAL ENERGY

Mechanical energy (ME) of an object is defined to be the sum of its potential energy and kinetic energy: $ME = KE + U$. Kinetic energy is always equal to $\frac{1}{2}mv^2$ while the expression for potential energy depends on the nature of the conservative force responsible for the potential energy.

$$ME = \frac{1}{2}mv^2 + U$$

If the only conservative force acting on an object is gravity, $U = m|g|y$ and the mechanical energy $\left(ME_g\right)$ of an object under the influence of gravity is given as

$$ME_g = \frac{1}{2}mv^2 + m|g|y$$

If the only conservative force acting on an object is the force due to a spring, $U = \frac{1}{2}kx^2$ and the mechanical energy (ME_e) of an object under the influence of a force due to a spring is given by

$$ME_e = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

If there are a number of conservative forces acting on an object then the potential energy is taken to be the sum of the potential energies due to each conservative force. For example, if an object is under the influence of gravity and force due to a spring, then $ME = \frac{1}{2}mv^2 + m|g|y + \frac{1}{2}kx^2$.

If all the forces with non-zero contribution to the work done on an object are conservative, then the net work done on the object is equal to the work done by the conservative forces; that is, $W_{net} = W_c$. But W_{net} and W_c are equal to ΔKE and $-\Delta U$ respectively. Therefore $\Delta KE = -\Delta U$ or $KE_f - KE_i = -(U_f - U_i)$. Rearranging this equation results in $KE_i + U_i = KE_f + U_f$ or $ME_i = ME_f$. This is a mathematical statement of the principle of conservation of mechanical energy. The *principle of conservation of mechanical energy* states that, if all the forces with non-zero contribution to the work done are conservative, then mechanical energy is conserved.

$$\frac{1}{2}mv_i^2 + U_i = \frac{1}{2}mv_f^2 + U_f$$

If gravity is the only conservative force with non-zero contribution to the work done acting, then

$$\frac{1}{2}mv_i^2 + m|g|y_i = \frac{1}{2}mv_f^2 + m|g|y_f$$

Or

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = -m|g|\left(y_f - y_i\right)$$

If the force due to a spring is the only conservative force with non-zero contribution to the work done acting, then

$$\frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2$$

Example: An object of mass 7 kg is sliding down a frictionless 10 m inclined plane. Calculate the speed of the object when it reaches the ground.

Solution: The forces acting are gravity (its weight) and the normal force. The normal force is perpendicular to the displacement and thus its contribution to the work done is zero. The only force with non-zero contribution to the work done is gravity which is conservative. Therefore, the principle of conservation of mechanical energy applies. Let (x_i, y_i) and (x_f, y_f) be the coordinates of the initial and final point respectively.

$m = 7$ kg; $v_i = 0$ (sliding); $d = 10$ m; $\theta = 30°$; $v_f = ?$

$$y_i - y_f = d\sin(\theta) = 10\sin(30°)\ \text{m} = 5\ \text{m} \Rightarrow y_f - y_i = -5\ \text{m}$$

$$\frac{1}{2}mv_f^{\,2} - \frac{1}{2}mv_i^{\,2} = -m|g|(y_f - y_i)$$

$$v_f^{\,2} = v_i^2 - |g|(y_f - y_i) = -9.8\times(-5)\ \text{m}^2/\text{s}^2 = 49\ \text{m}^2/\text{s}^2$$

$$v_f = 7\ \text{m/s}$$

Example: A pendulum of length ℓ is displaced by an angle θ and then let go. Calculate its speed at its lowest point.

Solution: The forces acting are the tension in the string and gravity. The tension is always perpendicular to its trajectory and thus the tension does not contribute to the work done. The only force that contributes to the work done is gravity which is conservative. Therefore the principle of conservation of mechanical energy applies. When the pendulum is displaced by an angle θ, the vertical distance between the object and the pivot changes from ℓ to $\ell\cos(\theta)$ Assuming the origin is at the pivot, $y_i = -\ell\cos(\theta)$ and $y_f = -\ell$ Therefore $y_f - y_i = -\ell - (-\ell\cos\theta) = -\ell(1-\cos\theta)$.

$v_i = 0;\ v_f = ?$

$$\frac{1}{2}mv_f^{\,2} - \frac{1}{2}mv_i^{\,2} = -m|g|\left(y_f - y_i\right) = -m|g|\left(-\ell\left(1-\cos\theta\right)\right) = m|g|\ell\left(1-\cos\theta\right)$$

$$v_f^{\,2} = 2\ell|g|\left(1-\cos\theta\right)$$

$$v_f = \sqrt{2\ell|g|\left(1-\cos\theta\right)}$$

Example: An object of mass 4 kg is attached to a spring of Hook's constant 200 N/m. The spring is then compressed by 4 cm and then let go on a horizontal frictionless surface. Calculate the speed of the object by the time it leaves the spring.

Solution: The forces acting are normal force and force due to a spring. The normal force does not contribute to the work done because it is perpendicular to the displacement. The only force that contributes to the work done is the force due to the spring which is conservative. Therefore the principle of conservation of mechanical energy applies.

$m = 4$ kg; $k = 200$ N/m; $x_i = 0.04$ m; $x_f = 0$ (relaxed position); $v_i = 0$ (let go); $v_f = ?$

$$\frac{1}{2}mv_i^{\,2} + \frac{1}{2}kx_i^{\,2} = \frac{1}{2}mv_f^{\,2} + \frac{1}{2}kx_f^{\,2}$$

$$kx_i^{\,2} = mv_f^{\,2}$$

$$v_f = x_i\sqrt{k/m} = 0.04\sqrt{\frac{200}{4}}\ \text{m/s} = 0.28\ \text{m/s}$$

Example: A 2 kg object is placed on a vertical spring of Hook's constant 600 N/m. The spring is then compressed further so that its compression from its relaxed position is 10 cm and then it is let go. How high will the object rise?

Solution: The only forces involved are gravity and the force due to the spring which are conservative. Therefore, mechanical energy is conserved. The potential energy of the object is the sum of gravitational and elastic potential energy.

$m = 2$ kg; $k = 600$ N/m; $v_i = 0$; $x_i = 0.1$ m; $v_f = 0$ (maximum height); $y_f - y_i = ?$

$$\frac{1}{2}mv_i^2 + m|g|y_i + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + m|g|y_f + \frac{1}{2}kx_f^2$$

$$m|g|y_i + \frac{1}{2}kx_i^2 = m|g|y_f$$

$$\left(y_f - y_i\right) = \frac{kx_i^2}{2m|g|} = \frac{600 \times 0.1^2}{2 \times 2 \times 9.8}\text{ m} = 0.153\text{ m}$$

8.7 WORK DONE BY NON-CONSERVATIVE FORCES

Examples of non-conservative forces are friction and air resistance. The net work done on an object is the sum of the work done by all conservative forces $\left(W_c\right)$ and non-conservative forces $\left(W_{nc}\right)$; That is, $W_{net} = W_c + Wn_c$. But W_{net} and W_c are equal to ΔKE and ΔU respectively. Therefore $W_{nc} = \Delta KE - \Delta U = \left(KE_f - KE_i\right) + \left(U_f - U_i\right) = \left(KE_f + U_f\right) - \left(KE_i + U_i\right)$. Work done by non-conservative force is equal to the change (loss) of mechanical energy of the object.

$$W_{nc} = \Delta ME = ME_f - ME_i$$

If the conservative force is gravitational force, then

$$W_{nc} = \left(\frac{1}{2}mv_f^2 + m|g|y_f\right) - \left(\frac{1}{2}mv_i^2 + m|g|y_i\right)$$

If the conservative force is force due to a spring, then

$$W_{nc} = \left(\frac{1}{2}mv_f^2 + \frac{1}{2}kx_f^2\right) - \left(\frac{1}{2}mv_i^2 + \frac{1}{2}kx_i^2\right)$$

Example: A 2 kg object is sliding down a 10 m 30° inclined plane. If it is found that its speed at the bottom is 5 m/s,

a) Calculate the work done by friction.
Solution: The forces acting are gravity, friction and normal force. The work done by normal force is zero because it is perpendicular to the displacement. Gravity is conservative. The only non-conservative force is friction. Therefore $W_{nc} = W_f$ (work done by friction).
$m = 2$ kg; $d = 10$ m; $\theta = 30°$; $v_i = 0$; $v_f = 5$ m/s; $W_f = ?$

$$y_i - y_f = d\sin(\theta) = 10\sin(30°)\ \text{m} = 5\ \text{m} \Rightarrow y_f - y_i = -5\ \text{m}$$

$$W_{nc} = W_f = \left(\frac{1}{2}mv_f^2 + m|g|y_f\right) - \left(\frac{1}{2}mv_i^2 + m|g|y_i\right) = \left(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2\right) + m|g|(y_f - y_i)$$

$$= \left(\frac{1}{2}\times 2\times 5^2 - 0\right)\ \text{J} + 2|9.8|(-5)\ \text{J} = -73\ \text{J}$$

b) Calculate the force of friction.

Solution: Friction is opposite to the displacement. Therefore the angle between friction and the displacement (θ_f) is 180°.

$$W_f = fd\cos(\theta_f)$$

$$-73\ \text{J} = f(10\ \text{m})\cos(180°)$$

$$f = 7.3\ \text{N}$$

Practice Quiz 8.2

Choose the best answer

1. Which of the following statements is correct.
   A) The work done by a conservative force depends on the path followed.
   B) The work done by a conservative force is equal to the negative of the change in potential energy of the object.
   C) The potential energy of an object depends on the speed of the object.
   D) The work done by a conservative force is equal to the change in the kinetic energy of the object.
   E) Friction is a conservative force.

2. If all the forces acting on an object are conservative, then
   A) The work done is equal to the change in the mechanical energy of the object.
   B) The kinetic energy of the object is conserved
   C) The mechanical energy of the object is conserved.
   D) The potential energy of the object is conserved
   E) The change in kinetic energy is equal to the change in potential energy

3. An object of mass *24.3* kg is located at the top of a *20* m tall building. Calculate the gravitational potential energy of the object with respect to a point *1.7* m above the top of the building.
   A) *694.765* J
   B) *404.838* J
   C) *556.424* J
   D) *-556.424* J
   E) *-404.838* J

4. Calculate the mechanical energy of an *8.4* kg object at a location where its potential energy is *10* J if its speed as it crosses this location is *19.7* m / s.
   A) *1073.75* J
   B) *2247.093* J
   C) *336.258* J
   D) *1470.533* J
   E) *1639.978* J

5. Under the influence of conservative forces only, an object is displaced from point A to point B. It's potential energy and kinetic energy at point A are respectively *5.231* J and *3.564* J. Its kinetic energy at point B is *1.856* J. Calculate its potential energy at point B.
   A) *9.565* J
   B) *12.817* J
   C) *5.295* J
   D) *4.067* J
   E) *6.939* J

6. An object of mass *11* kg is sliding down a friction less inclined plane of length *5* m that makes an angle of *10* deg. Calculate its speed just before it hits the ground.
   A) *4.125* m / s
   B) *6.073* m / s
   C) *0.768* m / s
   D) *4.555* m / s
   E) *6.548* m / s

7. A roller coaster extends to the ground from a height of *34* m (point A) and then rises to a height of *14* m (point B). An object of mass *8* kg starts at point A with a speed of *6* m / s. Assuming the roller coaster is friction less, calculate the speed of the object by the time it reaches point B.
   A) *3.94* m / s
   B) *33.038* m / s
   C) *20.688* m / s
   D) *24.558* m / s
   E) *13.649* m / s

8. An object of mass 1.25 kg is brought in contact with a spring of Hook's constant $180\ \mathrm{N/m}$ that is compressed by 0.1 m. If the spring is let go free to expand, calculate the speed by which the object will leave the spring at its relaxed position.
   A) $0.877\ \mathrm{m/s}$
   B) $0.272\ \mathrm{m/s}$
   C) $1.2\ \mathrm{m/s}$
   D) $1.662\ \mathrm{m/s}$
   E) $1.415\ \mathrm{m/s}$

9. A pendulum of length 6.3 m is displaced by an angle of 70° and then it is let go. Calculate its speed at its lowest point.
   A) $1.895\ \mathrm{m/s}$
   B) $9.014\ \mathrm{m/s}$
   C) $4.182\ \mathrm{m/s}$
   D) $7.114\ \mathrm{m/s}$
   E) $11.64\ \mathrm{m/s}$

10. A vertical spring of Hook's constant *1150* N / m is compressed by *0.21* m and an object of mass *0.53* kg is placed on top of it. How high will the object rise when the spring is released?
    A) *2.699* m
    B) *4.882* m
    C) *1.973* m
    D) *7.119* m
    E) *7.633* m

11. A roller coaster extends to the ground from a height of *33* m (point A) and then rises to a height of *21* m (point B). An object of mass *10* kg is released from point A. If its speed at point B is found to be *5* m / s, calculate the work done by friction.
    A) *-279.542* J
    B) *-1051* J
    C) *-1370.952* J
    D) *-1521.65* J
    E) *-1640.725* J

12. A spring of Hook's constant *250* N / m is compressed by *0.32* m on a horizontal surface; and then an object of mass *0.23* kg is brought in contact with it. When the spring is released, the speed of the object at the relaxed position of the spring is found to be *3.5* m / s. Calculate the work done by friction.
    A) *-13.972* J
    B) *-18.6* J
    C) *-11.391* J
    D) *-20.497* J
    E) *-8.669* J

9 MOMENTUM AND COLLISIONS

Your goal for this chapter is to learn about momentum and its use to solve collision problems.

Momentum $(\vec{p})$ of an object is a measure of the amount of motion an object has. It is defined to be the product of the mass and the velocity of the object.

$$\vec{p} = m\vec{v}$$

Unit of measurement of momentum is kg m/s.

Example: Obtain the momentum of a 5 kg object moving with a velocity of $(2\hat{\imath} + 3\hat{\jmath})$ m/s.

Solution:
$m = 5$ kg; $\vec{v} = (2\hat{i} + 3\hat{j})$ m/s; $\vec{p} = ?$

$$\vec{p} = m\vec{v} = 5(2\hat{i} + 3\hat{j}) \text{ kg m/s} = (10\hat{i} + 15\hat{j}) \text{ kg m/s}$$

The rate of change of momentum of an object is related with the force acting on the object. From Newton's second law, $\vec{F} = m\vec{a}$. But $\vec{a} = \frac{d\vec{v}}{dt}$ and $\vec{p} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt}$. Therefore, the relationship between force and momentum is that the rate of change of the momentum of an object is equal to the force acting on the object.

$$\vec{F} = \frac{d\vec{p}}{dt}$$

The average of the force $\left(\bar{\vec{F}}\right)$ during a time interval Δt obtained as $\bar{\vec{F}} = \frac{1}{\Delta t}\int_{t}^{t+\Delta t}\frac{d\vec{p}}{dt}dt$ is given as

$$\bar{\vec{F}} = \frac{\Delta\vec{p}}{\Delta t} = \frac{\vec{p}_f - \vec{p}_i}{\Delta t}$$

Impulse $(\vec{I})$ acting on an object is defined to be the product of the force acting on an object and the time interval during which the force is applied. If the force is constant, then $\vec{I} = \vec{F}\Delta t$. If an object is acted upon by a variable force $\vec{F}$ for a time interval Δt, then the impulse may be written as the product of the average of the force times the time interval or the integral of the force with time on the time interval.

$$\vec{I} = \bar{\vec{F}}\Delta t = \int_{t}^{t+\Delta t}\vec{F}dt$$

The unit of measurement for impulse is Ns. The relationship between force and momentum implies $d\vec{p} = \vec{F}dt$. Integrating this equation gives $\int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \vec{p}_f - \vec{p}_i = \Delta\vec{p} = \int_{t}^{t+\Delta t} \vec{F}dt$ which is equal to the impulse impacted on the object during the time interval. In other words the impulse of an object is equal to the change in momentum produced by the force during the time interval.

$$\vec{I} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

Since impulse (change of momentum) is equal to the integral of force over time, it can be obtained from a graph of force versus time as the area enclosed between the force versus time curve and the time axis.

Example: The force acting on an object of mass 2 kg varies with time according to the Equation $\vec{F} = \left(-at^2 + bt\right)\hat{i}$ where $a = 1\text{ N/m}^2$ and $b = 2\text{ N/s}$.

a) Obtain the change in its momentum in the first 2 seconds.
Solution:
$m = 2\text{ kg};\ t_i = 0;\ t_f = 2\text{ s};\ \Delta\vec{p} = ?$

$$\Delta\vec{p} = \int_{t_i}^{t_f} \vec{F}\,dt = \hat{i}\int_{0}^{2\text{ s}} \left(-at^2 + bt\right)dt = \hat{i}\left(-\frac{at^3}{3} + \frac{bt^2}{2}\right)\Bigg|_{0}^{2\text{ s}} = \hat{i}\left(-\frac{2^3}{3} + \frac{2\times 2^2}{2}\right)\text{ N} = \frac{4}{3}\hat{i}\text{ kg m/s}$$

b) Calculate the impulse acting on it during the first 2 seconds.

Solution:

$\vec{I} = ?$

$$\vec{I} = \Delta\vec{p} = \frac{4}{3}\hat{i} \text{ Ns}$$

c) If its initial velocity is 4 m/s, calculate its velocity after 2 seconds.

Solution:

$\vec{v}_i = 4\hat{i}$ m/s; $v_f = ?$

$$\Delta\vec{p} = \vec{p}_f - \vec{p}_i = m\vec{v}_f = m\vec{v}_i$$

$$\frac{4}{3}\hat{i} \text{ kg m/s} = (2 \text{ kg})\vec{v}_f - 2\times 4 \text{ kg m/s}$$

$$\vec{v}_f = \frac{14}{3}\hat{i} \text{ m/s}$$

Example: An object of mass 5 kg hits a wall with a velocity of 10 m/s 37° north of east and bounces back with a velocity of 8 m/s 53° west of north.

a) Calculate the change in its momentum.

Solution:

$m = 5$ kg; $v_i = 10$ m/s; $\theta_i = 37°$; $v_f = 8$ m/s; $\theta_f = 90° + 53° = 143°$; $\Delta\vec{p} = ?$

$$\vec{v}_i = v_i \cos(\theta_i)\hat{i} + v_i \sin(\theta_i)\hat{j} = \left(8\hat{i} + 6\hat{j}\right) \text{ m/s}$$

$$\vec{v}_f = v_f \cos(\theta_f)\hat{i} + v_f \sin(\theta_f)\hat{j} = \left(-6.4\hat{i} + 4.8\hat{j}\right) \text{ m/s}$$

$$\Delta\vec{p} = m(\vec{v}_f - \vec{v}_i) = 5\left(\left(-6.4\hat{i} + 4.8\hat{j}\right) - \left(8\hat{i} + 6\hat{j}\right)\right) \text{ kg m/s} = \left(-72\hat{i} - 6\hat{j}\right) \text{ kg m/s}$$

b) Calculate the impulse impacted by the wall on the object.

Solution:

$\vec{I} = ?$

$$\vec{I} = \Delta\vec{p} = \left(-72\hat{i} - 6\hat{j}\right) \text{ Ns}$$

c) If the object was in contact with the wall for 0.2 seconds, calculate the average force exerted by the ball on the object.

Solution:

$\Delta t = 0.2$ s; $\bar{\vec{F}} = ?$

$$\bar{\vec{F}} = \frac{\Delta\vec{p}}{\Delta t} = \frac{\left(-72\hat{i} - 6\hat{j}\right)}{0.2}\text{ N} = \left(-360\hat{i} - 30\hat{j}\right)\text{ N}$$

9.1 CONSERVATION OF MOMENTUM

If the net force acting on an object is zero, then $\vec{F} = \frac{d\vec{p}}{dt} = 0$ which implies $d\vec{p} = 0$. Therefore $\int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \vec{p}_f - \vec{p}_i = 0$ or $\vec{p}_f = \vec{p}_i$. *The principle of conservation of momentum* states that if the net force acting on an object is zero, then the momentum of the object is conserved.

$$\text{If } \vec{F}_{net} = 0 \text{ then } \vec{p}_f = \vec{p}_i$$

The momentum of an isolated system of particles (an isolated system is a system that does not interact with other systems) is conserved because all the forces involved are internal forces, and the sum of all internal forces is zero because for every action there is a reaction.

The momentum of two colliding objects is conserved if they are treated as a single system because the net force acting on the system is zero. The net force is zero because the forces exerted on each other are action-reaction forces. That is, $\vec{F}_{AB}$ and $\vec{F}_{BA}$ are the forces exerted by two colliding objects A and B on each other, then $\vec{F}_{net} = \vec{F}_{AB} + \vec{F}_{BA} = 0$ because $\vec{F}_{BA} = -\vec{F}_{AB}$. If $\vec{p}_{1i}$ If $\vec{p}_{1i}$ and $\vec{p}_{2i}$ are the momentums of two colliding objects before collision and $\vec{p}_{1f}$ and $\vec{p}_{2f}$ are their momentums after collision, then

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

Or

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

9.2 ONE DIMENSIONAL COLLISION

One dimensional collision is collision in a straight line. If a coordinate system where the x-axis lies along the line of collision is used, then $\vec{v}_{1i} = v_{1ix}\hat{i}$, $\vec{v}_{1f} = v_{1fx}\hat{i}$, $\vec{v}_{2i} = v_{2ix}\hat{i}$, and $\vec{v}_{2f} = v_{2fx}\hat{i}$ and the equation for conservation of momentum in terms of components becomes

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

In this equation, the velocities can be positive or negative because they are components. The velocity is taken to be positive if it is to right and negative if it is to the left. For a one dimensional motion, since there are two possible directions, a component in terms of the magnitude of the velocity as $v_x = \pm v_i$ where the "+" sign is chosen if the motion is to the right and the "–" sign if the motion is to the left. Therefore the equation of conservation of momentum can be written in terms of magnitudes as

$$m_1\left(\pm v_{1i}\right)+m_2\left(\pm v_{2i}\right)=m_1\left(\pm v_{1f}\right)+m_2\left(\pm v_{2f}\right)$$

Example: An object of mass 10 kg going to the right with a speed of 5 m/s collides with a 2 kg object moving with a speed of 3 m/s to the left. After the collision, the 2 kg object moves to the right with a speed of 20 m/s. Calculate the velocity of the 10 kg object after collision.

Solution:
$m_1 = 10$ kg; $v_{1i}x = 5$ m/s; $m_2 = 2$ kg; $v_{2i}x = -3$ m/s ; $v_{2f} = 20$ m/s; $v_{1fx} = ?$

$$m_1\left(v_{1ix}\right)+m_2\left(v_{2ix}\right)=m_1\left(v_{1fx}\right)+m_2\left(v_{2fx}\right)$$

$$10\times 5+2\times -3=10\left(v_{1fx}\right)+2\times 20$$

$$v_{1fx} = 0.4 \text{ m/s}$$

9.3 COMPLETELY INELASTIC COLLISIONS

A *completely inelastic collision* is a collision where the two objects stick together after collision. After collision, the two objects will have the same velocity. Let their common velocity after collision be denoted by $\vec{v}$. Then, $\vec{v}_{1f} = \vec{v}_{2f} = \vec{v}$ and the equation of conservation of momentum for a completely elastic collision becomes

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}$$

For a one dimensional collision, this can be written as

$$m_1 v_{1ix} + m_2\vec{v}_{2ix} = (m_1 + m_2)\vec{v}_x$$

Or

$$m_1(\pm v_{1i}) + m_2(\pm v_{2i}) = (m_1 + m_2)(\pm v)$$

For completely inelastic collision, kinetic energy is not conserved because some of the energy will be converted to heat or internal energy. The lost kinetic energy (ΔKE) is equal to the difference between the before and after kinetic energies.

$$\Delta kE = KE_f - KE_i = \frac{1}{2}(m_1 + m_2)v^2 - \left(\frac{1}{2}m_1 v_{1i}{}^2 + \frac{1}{2}m_2 v_{2i}{}^2\right)$$

Example: A 6 kg object moving to the right with a speed of 5 m/s collides with a 4 kg object moving in the same direction with a speed of 3 m/s. If the collision is completely inelastic

a) Calculate their velocity after collision.
   Solution:
   $m_1 = 6\text{ kg};\ v_{1ix} = 5\text{ m/s};\ m_2 = 4\text{ kg};\ v_{2ix} = 3\text{ m/s};\ v_x = ?$

$$m_1(v_{1ix}) + m_2(v_{2ix}) = (m_1 + m_2)(v_x)$$

$$(6\times 5 + 4\times 3)\text{ kg m/s} = v_x(6 + 4)\text{ kg}$$

$$v_x = 4.2\text{ m/s}$$

b) Calculate the kinetic energy lost during the collision.
Solution:

$$\Delta kE = KE_f - KE_i = \frac{1}{2}(m_1 + m_2)v^2 - \left(\frac{1}{2}m_1 v_{1i}{}^2 + \frac{1}{2}m_2 v_{2i}{}^2\right)$$

$$\frac{1}{2}(6+4)(4.2)^2 \text{ J} - \left[\frac{1}{2}(6)(5)^2 + \frac{1}{2}(4)(3)^2\right] \text{ J}$$

$$\Delta KE = 88.2 \text{ J} - 93 \text{ J} = -4.8 \text{ J}$$

9.4 THE BALLISTIC PENDULUM

The *ballistic pendulum* is a pendulum used to measure the speed of a bullet. Suppose a bullet of mass m_b and speed v_b is fired horizontally into a pendulum of mass m_p which was initially at rest. And suppose the bullet is embedded in the pendulum and the bullet-pendulum system moves with a speed of after collision. Since the collision is completely inelastic $m_b v_b = (m_b + m_p)v$ or

$$v_b = \frac{(m_p + m_b)v}{m_b}$$

If the pendulum rises by a height of h, the kinetic energy at the bottom will be converted to potential energy and $\frac{1}{2}(m_b + m_p)v^2 = (m_b + m_p)|g|h$ which implies $v = \sqrt{2|g|h}$. Therefore, if the height to which the pendulum rises is measured, the speed of the bullet can be calculated from the formula

$$v_b = \frac{m_p + m_b}{m_b}\sqrt{2|g|h}$$

Practice Quiz 9.1

Choose the best answer

1. Momentum of an object is
   A) Measure of the amount of motion an object has.
   B) measure of the amount of kinetic energy an object has
   C) measure of the amount of potential energy an object has
   D) measure of the amount of mechanical energy an object has
   E) is a measure of the acceleration of an object

2. Calculate the speed of an object of mass *0.24* kg moving with a momentum of *2.4* kg m / s.
    A) *13.289* m / s
    B) *15.395* m / s
    C) *11.438* m / s
    D) *10* m / s
    E) *2.495* m / s

3. Calculate the kinetic energy of an object of mass *14.74* kg if its momentum is *40* kg m / s.
    A) *6.062* J
    B) *92.549* J
    C) *54.274* J
    D) *43.143* J
    E) *86.833* J

4. An object of mass *0.175* kg moving to the right with a speed of *17* m / s hits a wall and bounces back to the left with a speed of *9.3* m / s. Calculate the change in its momentum.
   A) *-0.885* kg m / s
   B) *-1.67* kg m / s
   C) *-7.671* kg m / s
   D) *-5.69* kg m / s
   E) *-4.603* kg m / s

5. A ball of mass *0.16* kg hit a wall with a speed of *18.2* m / s making an angle of *42*° with the vertical-down. It bounced back with a speed of *9.2* m / s making an angle of *43*° with the vertical-up. Calculate the magnitude of its change of momentum.
   A) *1.788* kg m / s
   B) *1.469* kg m / s
   C) *3.146* kg m / s
   D) *4.799* kg m / s
   E) *5.254* kg m / s

6. An object was acted upon by a force of *18.33* for *0.87* seconds. Calculate the change in its momentum.
   A) *10.057* kg m / s
   B) *22.795* kg m / s
   C) *7.75* kg m / s
   D) *26.535* kg m / s
   E) *15.947* kg m / s

7. The force acting on a particle of mass *7.4* kg varies with time according to the equation $F = -10t^2 + 80t - 159.1$ If its initial speed at $t = 3.7$ s is *5.3* m / s, calculate its speed at $t = 4.3$ s.
   A) *5.349* m / s
   B) *9.29* m / s
   C) *2.387* m / s
   D) *6.268* m / s
   E) *4.592* m / s

8. An object of mass *16.84* kg moving with a speed of *27* m / s to the right collides with an object of mass *3* kg moving with a speed of *10* m / s to the left. After collision, the *3* kg object moves to the right with a speed of *8* m / s to the right. Calculate the velocity of the *16.84* kg object after collision.
   A) *23.793* m / s
   B) *21.016* m / s
   C) *26.201* m / s
   D) *44.519* m / s
   E) *12.392* m / s

9. An object of mass *4.23* kg moving with a speed of *24* m / s to the right collides with an object of mass *3* kg moving with a speed of *4* m / s to the left. If the collision is completely inelastic, calculate their speed after collision.
   A) *10.679* m / s
   B) *5.822* m / s
   C) *20.143* m / s
   D) *23.074* m / s
   E) *12.382* m / s

10. After a bullet of mass *0.067* kg is fired into a ballistic pendulum of mass *4* kg, the bullet is embedded in the pendulum and the pendulum rose to a height of *0.45* m. Calculate the speed with which the bullet was fired into the ballistic pendulum.
   A) *103.207* m / s
   B) *159.415* m / s
   C) *270.425* m / s
   D) *305.489* m / s
   E) *180.274* m / s

9.5 COMPLETELY ELASTIC COLLISIONS

A *completely elastic collision* is a collision where not only momentum but also kinetic energy is conserved. Therefore in this case there are two conservation equations. Therefore for a completely elastic collision the following two equations apply.

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$\frac{1}{2}m_1{v_{1i}}^2 + \frac{1}{2}m_2{v_{2i}}^2 = \frac{1}{2}m_1{v_{1f}}^2 + \frac{1}{2}m_2{v_{2f}}^2$$

For one dimensional collisions, these equations the second equation can be simplified by combining both equations. Using a coordinate system where the x-axis is along the line of collision, for one dimensional motion, these equations become $m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$ and $\frac{1}{2} m_1 v_{1ix}{}^2 + \frac{1}{2} m_2 v_{2ix}{}^2 = \frac{1}{2} m_1 v_{1fx}{}^2 + \frac{1}{2} m_2 v_{2fx}{}^2$. Putting m_1 terms on the left side and m_2 terms on the other side, these equation turn into $m_1\left(v_{1ix} - v_{1fx}\right) = m_2\left(v_{2fx} - v_{2ix}\right)$ and $m_1\left(v_{1ix}^2 - v_{1fx}^2\right) = m_2\left(v_{2fx}^2 - v_{2ix}^2\right)$. Factorizing the difference between squares of the second term, the second equation can also be written as $m_1\left(v_{1ix} + v_{1fx}\right)\left(v_{1ix} - v_{1fx}\right) = m_2\left(v_{2fx} + v_{2ix}\right)\left(v_{2fx} - v_{2ix}\right)$. Now dividing the equation obtained from the conservation of kinetic energy by the equation obtained from the conservation of momentum the equation $v_{1ix} - v_{2ix} = -\left(v_{1fx} - v_{2fx}\right)$. Therefore, for a one dimensional elastic collision, the following simpler equations can be used.

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$v_{1ix} - v_{2ix} = -\left(v_{1fx} - v_{2fx}\right)$$

For a one dimensional elastic collision, if the initial velocities are known, the final velocities can be predicted.

Example: An object of mass 4 kg moving to the right with a speed of 10 m/s collides with a 2 kg object moving in the same direction with a speed of 5 m/s. If the collision is completely elastic, calculate their speeds after collision.

Solution:
$m_1 = 4$ kg; $v_{1ix} = 10$ m/s; m_2=2 kg; $v_{2ix} = 5$ m/s; $v_{1fx} = ?$; $v_{2fx} = ?$

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$(4\times 10 + 2\times 5)\ \text{kg m/s} = (4\ \text{kg})v_{1fx} + (2\ \text{kg})v_{2fx}$$

$$4v_{1fx} + 2v_{2fx} = 50\ \text{m/s}$$

$$v_{1ix} - v_{2ix} = -(v_{1fx} - v_{2fx})$$

$$(10-5)\ \text{m/s} = -(v_{1fx} - v_{2fx})$$

$$(v_{1fx} - v_{2fx}) = -5\ \text{m/s}$$

$$4v_{1fx} + 2v_{2fx} = 50\ \text{m/s and } (v_{1fx} - v_{2fx}) = -5\ \text{m/s} \Rightarrow v_{1fx} = \frac{20}{3}\ \text{m/s and } v_{2fx} = \frac{35}{3}\ \text{m/s}$$

9.6 TWO DIMENSIONAL (GLANCING) COLLISIONS

A two dimensional collision is a collision where different directions in a plane are involved for the velocities. Therefore in this case the vector form of the conservation of momentum should be used.

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

With $\vec{v}_{1i} = v_{1ix}\hat{i} + v_{1iy}\hat{j}$, $\vec{v}_{1f} = v_{1fx}\hat{i} + v_{1fy}\hat{j}$, $\vec{v}_{2i} = v_{2ix}\hat{i} + v_{2iy}\hat{j}$, and $\vec{v}_{2f} = v_{2fx}\hat{i} + v_{2fy}\hat{j}$, this vector equation can be written as two algebraic component equations.

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$
$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

Example: An object of mass 4 kg going east with a speed of 4 m/s collided with a 3 kg object going north with a speed of 2 m/s. After collision the 4 kg object movies with a speed of 3 m/s making an angle of 53° with the horizontal.

a) Determine the x and y components of the velocity of the 3 kg object after the collision.
Solution:
$m_1 = 4$ kg; $v_{1i} = 4$ m/s; $\theta_{1i} = 0$; v_{1f}=3 m/s; θ_{1f}=53°; $m_2 = 3$ kg; $v_{2i} = 2$ m/s; $\theta_{2i} = 90°$; $v_{2fx} = ?$; $v_{2fy} = ?$

$$\vec{v}_{1i} = \left(4\cos\left(0\right)\hat{i} + 4\sin\left(0\right)\hat{j}\right) \text{ m/s} \Rightarrow v_{1ix} = 4 \text{ m/s and } v_{1iy} = 0$$

$$\vec{v}_{1i} = \left(4\cos\left(0\right)\hat{i} + 4\sin\left(0\right)\hat{j}\right) \text{ m/s} \Rightarrow v_{1ix} = 4 \text{ m/s and } v_{1iy} = 0$$

$$\vec{v}_{1f} = \left(3\cos\left(53°\right)\hat{i} + 3\sin\left(53°\right)\hat{j}\right) \text{ m/s} \Rightarrow v_{1fx} = 1.8 \text{ m/s and } v_{1fy} = 2.4 \text{ m/s}$$

$$\vec{v}_{2i} = \left(2\cos\left(90°\right)\hat{i} + 2\sin\left(90°\right)\hat{j}\right) \text{ m/s} \Rightarrow v_{2ix} = 0 \text{ and } v_{2iy} = 2 \text{ m/s}$$

$$\left(4\times4+3\times0\right) \text{ kg m/s} = \left(4\times1.8\right) \text{ kg m/s} + \left(3 \text{ kg}\right)v_{2fx}$$

$$v_{2fx} = 2.93 \text{ m/s}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

$$\left(4\times0+3\times2\right) \text{ kg m/s} = \left(4\times2.4\right) \text{ kg m/s} + \left(3 \text{ kg}\right)v_{2fy}$$

$$v_{2fy} = -1.2 \text{ m/s}$$

b) Determine the magnitude and direction of the velocity of the 3 kg object after collision.
Solution:
$v_{2f} = ?$; $\theta_{2f} = ?$

$$v_{2f} = \sqrt{{v_{2fx}}^2 + {v_{2fy}}^2} = \sqrt{2.93^2 + \left(-1.2\right)^2} \text{ m/s} = 3.17 \text{ m/s}$$
$$\theta_{2f} = \tan^{-1}\left(\frac{v_{2fy}}{v_{2fx}}\right) = \tan^{-1}\left(\frac{-1.2}{2.93}\right) = -22.27°$$

9.7 CENTER OF MASS

Center of Mass of a System of Particles: The *center of mass* $\left(\vec{r}_{CM}\right)$ of a system of particles is an average of the position vectors of all the particles of the system weighted by their masses.

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

Where $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots$ are the position vectors of the particles and $m_1, m_2, m_3, \ldots$ are the masses of the particles. Since the x and y components of a position vector are the x and y coordinates respectively, in component form, this becomes

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

$$y_{CM} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

Where (x_{CM}, y_{CM}) is the coordinate of the center of mass and $(x_1, y_1), (x_2, y_2), \ldots$ are the coordinates of the particles.

Example: A system is comprised of 3 particles. Particle 1 has a mass of 3 kg and is located at $(-2,5)$ m. Particle 2 has a mass of 4 kg and is +located at $(3,6)$ m. Particle 3 has a mass of 2 kg and is located at $(4,-5)$ m. Determine the location of the center of mass of the system.

Solution:

$m_1 = 3$ kg; $(x_1, y_1) = (-2, 5)$ m; $m_2 = 4$ kg; $(x_2, y_2) = (3, 6)$ m; $m_3 = 2$ kg; $(x_3, y_3) = (4, -5)$ m

$(x_{CM}, y_{CM}) = ?$

$$x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{3\times -2 + 4\times 3 + 2\times 4}{3+4+2} \text{ m} = \frac{14}{9} \text{ m}$$

$$y_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{3\times 5 + 4\times 6 + 2\times -5}{3+4+2} \text{ m} = \frac{29}{9} \text{ m}$$

$$(x_{CM}, y_{CM}) = \left(\frac{14}{9}, \frac{29}{9}\right) \text{ m}$$

Center of Mass of Solid Objects: To find the center of mass of a solid, we can start by assuming the solid is divided into small parts whose masses are $\Delta m_1, \Delta m_2, \ldots$ and then taking the limiting value as the masses of the small parts approach zero. $\vec{r}_{CM} = \lim\limits_{\Delta m_i \to 0} \frac{\sum_i \Delta m_i \vec{r}_i}{\sum_i \Delta m_i} = \lim\limits_{\Delta m_i \to 0} \frac{1}{M} \sum_i \vec{r}_i \Delta m_i$ (Where $M = \sum_i \Delta m_i$ is the total mass of the solid). But this is equal to the integral of $\vec{r}dm$ divided by the total mass.

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}dm$$

Since $\vec{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j}$ and $\vec{r} = x\hat{i} + y\hat{j}$, this also may be written in component form as

$$x_{cm} = \frac{1}{M}\int x\, dm$$
$$y_{cm} = \frac{1}{M}\int y\, dm$$

This integrals can be converted from integration over mass to integration over position coordinates using the definition of density. If the mass dm is contained in a volume element dV, density (ρ) at the location of dm is defined as $\rho = \frac{dm}{dV}$. Now the integrals can be written in terms of density as

$$x_{CM} = \frac{1}{M}\int x\,\rho dV$$
$$y_{cm} = \frac{1}{M}\int y\rho dV$$

For one dimensional (linear) and two dimensional (areal) problems, the formulae for density should be modified to $\lambda = \frac{dm}{dx}$ and $\sigma = \frac{dm}{dA}$ respectively. (*dx* and *dA* are linear and areal path elements respectively).

Example: Find the location of the center of mass of uniform rod of length *L* and mass *M*.

Solution: Let's use a coordinate system where the x-axis is along the rod and the origin is at the left end of the rod. Let *dx* be an arbitrary path element on the rod (whose mass is *dm*) located at a distance *x* from the origin. Since the rod is uniform, $\lambda = \frac{dm}{dx} = \frac{M}{L}$. Therefore the center of mass is given as $x_{CM} = \frac{1}{M}\int_0^L x\lambda dx = \frac{1}{M}\int_0^L x\left(\frac{M}{L}\right)dx = \frac{L}{2}$

Example: Find the location of the center of mass of a uniform rectangular plate of sides *a* and *b*; and mass *M*.

Solution: Let's use a coordinate system where the origin lies at the lower left corner of the rectangular plate and the x-axis and the y-axis lie along the sides of length *a* and *b* respectively. Since it is uniform the areal density is a constant and is equal to the ratio between the total mass and the total area $(A = ab)$. To calculate the x-coordinate of the center of mass (x_{CM}), let the small mass element of mass *dm* be a vertical strip of width *dx* and height *b* located at an arbitrary perpendicular distance *x* from the y-axis where the value of *x* varies from 0 to a. Since the density is a constant, $\frac{dm}{bdx} = \frac{M}{ab}$ or $dm = \frac{M}{a}dx$. Therefore the x-coordinate of the center of mass is given as $x_{CM} = \frac{1}{M}\int xdm = \frac{1}{M}\int_{x=0}^{x=a} x\left(\frac{M}{a}dx\right) = \frac{a}{2}$. By a similar process, it can be shown that $y_{CM} = \frac{b}{2}$. Thus the center of mass of the rectangular plate is located at $\vec{r}_{CM} = (x_{CM}, y_{CM}) = \left(\frac{a}{2}, \frac{b}{2}\right)$ as expected.

Example: A rod extends from $x = 0$ to $x = 2$ m along the x-axis. The linear density of the rod depends on *x* according to the equation $\lambda = ax + b$ where $a = 2\text{ kg/m}^2$ and $b = 1\text{ kg/m}$. Find the location of its center of mass.

Solution: $\lambda = \frac{dm}{dx} = ax + b \Rightarrow dm = (ax + b)dx$. Therefore $M = \int dm = \int_{x=0}^{x=2\text{ m}} (ax + b)dx = 6\text{ kg}$. And $x_{CM} = \frac{1}{M}\int xdm = \frac{1}{6\text{ kg}}\int_{x=0}^{x=2\text{ m}} x(ax + b)dx = 1.22\text{ m}$.

Velocity of the Center of Mass: The velocity (v_{CM}) of the center of mass of a system of particles is equal to the rate of change of the position vector of the center of mass with time. $\frac{d\vec{r}_{cm}}{dt} = \vec{v}_{cm} = \frac{m_1 \frac{d\vec{r}_1}{dt} + m_2 \frac{d\vec{r}_2}{dt} + \ldots}{M}$. That is, the velocity of the center of mass is the weighted average of the velocities of the particles weighted by their masses.

$$\vec{v}_{cm} = \frac{1}{M}(m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots) = \frac{1}{M}\sum_i m_i\vec{v}_i$$

Example: Three particles of masses 1 kg, 2 kg, and 3 kg are moving with velocities $\left(-2\hat{i}+3\hat{j}\right)$ m/s, $\left(5\hat{i}-7\hat{j}\right)$ m/s, and $-4\hat{i}$ m/s, respectively. Determine the velocity of the center of mass.

Solution:
$m_1 = 1$ kg; $v_1 = \left(-2\hat{i}+3\hat{j}\right)$ m/s; $m_2 = 2$ kg; $v_2 = \left(5\hat{i}-7\hat{j}\right)$ m/s; $m_3 = 3$ kg; $v_3 = -4\hat{i}$ m/s; $\vec{v}_{CM} = ?$

$$M = m_1 + m_2 + m_3 = (1+2+3)\,\text{kg} = 6\text{ kg}$$

$$\vec{v}_{cm} = \frac{1}{6}\left[1\left(-2\hat{i}+3\hat{j}\right)+2\left(5\hat{i}-7\hat{j}\right)+3\left(-4\hat{i}\right)\right]\text{ m/s} = \left(-\frac{2}{3}\hat{i}-\frac{11}{6}\hat{j}\right)\text{ m/s}$$

Acceleration of the center of mass of a system of particles: The acceleration of the center of mass of a system of particles is equal to the rate of change of the velocity of the center of mass with time. $\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M}\left(m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} + m_3\frac{d\vec{v}_3}{dt} + \ldots\right)$. That is, the acceleration of the center of mass is equal to the average of the accelerations of the particles weighted by their masses.

$$\vec{a}_{CM} = \frac{1}{M}\left(m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots\right) = \frac{1}{M}\sum_i \vec{a}_i m_i$$

Example: The center of mass of the system of two particles of masses 10 kg and 20 kg is moving with an acceleration of $\left(4\hat{i} - 3\hat{j}\right)$ m/s^2. If the 10 kg object is moving with an acceleration of $6\hat{j}$ m/s^2, find the acceleration of the 20 kg object.

Solution:

$m_1 = 10$ kg; $\vec{v}_1 = 6\hat{j}$ m/s^2; $\vec{a}_{CM} = \left(4\hat{i} - 3\hat{j}\right)$ m/s^2; $m_2 = 20$ kg; $\vec{v}_2 = ?$

$$M = m_1 + m_2 = (10+20) \text{ kg} = 30 \text{ kg}$$

$$M\vec{a}_{CM} = m_1\vec{a}_1 + m_2\vec{a}_2$$

$$30\left(4\hat{i} - 3\hat{j}\right) \text{ N} = 10\left(6\hat{j}\right) \text{ N} + (20 \text{ kg})\vec{a}_2$$

$$\vec{a}_{CM} = \left(6\hat{i} - 7.5\hat{j}\right) \text{ m/s}^2$$

Net force acting on system of particles: The internal forces exerted among the particles themselves will add up to zero because they are action reaction pairs. That is, for every force exerted by one particle on another there will be a reaction force equal but opposite in direction. Therefore the net force acting on a system of particles is equal to the net external force (forces exerted by particles outside the system). The net external force is equal to the sum of all the external forces exerted on the particles in the system. That is, $\vec{F}_{ext} = m_a\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots$. But $m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots = M\vec{a}_{cm}$. The external force acting on a system of particles is equal to the product of the total mass and the acceleration of the center of mass. In other words, the external force can be calculated as if the entire mass of the system is located on the center of mass.

$$\vec{F}_{ext} = M\vec{a}_{cm}$$

Practice Quiz 9.2

Choose the best answer

1. If a collision is a completely inelastic collision, then
   A) The kinetic energy of the colliding objects increases during the collision.
   B) The kinetic energy of the colliding objects remains the same after collision.
   C) After collision, the colliding objects will move in opposite directions.
   D) the colliding objects will have the same velocity after collision
   E) the speeds after collision remain the same with the speeds before collision

2. An object of mass *17.22* kg going to the right with a speed of *24* m / s collides with a *10.4* kg object going in the same direction with a speed of *5* m / s. If the collision is completely elastic, calculate the speed of the *10.4* kg object after collision.
   A) *53.781* m / s
   B) *28.692* m / s
   C) *21.373* m / s
   D) *48.66* m / s
   E) *39.803* m / s

3. An object of mass *15.11* kg going to the right with a speed of *18* m / s collides with a *14.9* kg object going to the left with a speed of *20* m / s. If the collision is completely elastic, calculate the speed of the *14.9* kg object after collision.
   A) *28.942* m / s
   B) *18.266* m / s
   C) *14.149* m / s
   D) *34.208* m / s
   E) *4.692* m / s

4. An object of mass *19* kg going to the right with a speed of *22* m / s collides with a *20* kg object at rest. After collision the *19* kg object moves with a speed of *15* m / s making an angle of *50* degree with the horizontal-right. Calculate the x-component of the velocity of the *20* kg object after collision.
   A) *8.592* m / s
   B) *17.944* m / s
   C) *21.669* m / s
   D) *11.74* m / s
   E) *9.984* m / s

5. An object of mass *5* kg going to towards north with a speed of *30* m / s collides with a(n) *12* kg object going east with a speed of *37* m / s. After collision the *5* kg object moves with a speed of *15* m / s making an angle of *60* degree with the horizontal-right. Calculate the magnitude of the velocity of the *12* kg object after collision.
   A) *34.608* m / s
   B) *4.924* m / s
   C) *65.063* m / s
   D) *12.685* m / s
   E) *18.798* m / s

6. The center of mass of two particles of masses *6* kg and *13.1* kg is located at the point (*-3, 0*) m. The *6* kg particle is located at the point (*2, -2*) m. Find the location of the *13.1* kg particle.
   A) (*-1.591*, *1.234*) m
   B) (*-5.29*, *0.916*) m
   C) (*-1.591*, *1.501*) m
   D) (*-1.591*, *0.916*) m
   E) (*-5.29*, *1.234*) m

7. A rod extends from $x = 0$ to $x = 4.5$ m along the x-axis. The linear density of the rod varies with x according the equation $\lambda = 1/x^{0.3} + 1.3$ Find the location of the center of mass of this rod.
   A) *1.831* m
   B) *2.087* m
   C) *0.619* m
   D) *2.944* m
   E) *1.586* m

8. The center of mass of a system of two particles is going with a velocity of *2.6* m / s *60*° south of east. The masses of the particles are *5* kg and *10* kg. The *5* kg particle is moving with a velocity of *4.5* m/s *50*° north of west. Calculate the direction of the velocity of the *10* kg particle.
   A) *-91.956*°
   B) *-48.22*°
   C) *-38.301*°
   D) *-102.029*°
   E) *-56.345*°

9. The position vectors of a *2* kg and a *4* kg particles vary with time according the equations $\boldsymbol{r} = 8.2t\,\boldsymbol{i} + 3.4t^2\,\boldsymbol{j}$ and $\boldsymbol{r} = 4.5t^3\,\boldsymbol{i} + 5.6t\,\boldsymbol{j}$ respectively. Calculate the magnitude of the velocity of the center of mass of the particles after *8.2* seconds.
   A) *110.946* m / s
   B) *1029.841* m / s
   C) *374.058* m / s
   D) *472.096* m / s
   E) *608.303* m / s

10. A system comprises of *5* particles of masses *2* kg, *7* kg, *4* kg, *1* kg and *8* kg. If the center of mass of the particles changed its speed uniformly from *3.4* m / s to *18.9* m / s in *3.41* seconds, calculate the net external force acting on the system of particles.
   A) *156.003* N
   B) *100* N
   C) *32.398* N
   D) *71.914* N
   E) *127.77* N

10 ROTATION OF A RIGID OBJECT ABOUT A FIXED AXIS

Your goal for this chapter is to learn about the relationships among rotation motion variables.

10.1 ANGLES

An angle is a measure of the inclination between two lines. There are two units of measurement for angles. They are the degree and the radian. A *degree* is defines to be $\left(\frac{1}{360}\right)^{\text{th}}$ of a complete circle. Therefore one revolution is equal to 360 degrees. A *radian* is defined to be a central angle that subtends an arc-length equal to the radius. Generally the radian measure of a central angle (θ) is defined to be the ratio of the arc-length (s) it subtends to the radius (r).

$$\theta = \frac{s}{r}$$

A radian is unit less. The radian measure of one revolution is 2π because for one revolution $s = 2\pi r$ (circumference). Using the fact that one revolution is equal to 360 degrees and 2π radians, the following relationships between a degree and a radian can be obtained.

$$\text{deg} = \frac{\pi}{180}\text{rad or deg} = \frac{180}{\pi}\text{deg}$$

Or

$$\frac{\theta}{\text{rad}} = \left(\frac{\pi}{180}\right)\frac{\theta}{\text{deg}} \text{ or } \frac{\theta}{\text{deg}} = \left(\frac{180}{\pi}\right)\frac{\theta}{\text{rad}}$$

Example: Calculate the radian measure of a central angle that subtends an arc length of 2 cm in a circle of radius 4 cm.

Solution:

$s = 4$ cm; $r = 2$ cm; $\theta = ?$

$$\theta = \frac{s}{2} = \frac{4}{2} = 2 \text{ rad}$$

Example: Convert

a) 120° to radians.
Solution:

$$120 \text{ deg} = 120\left(\frac{\pi}{180} \text{ rad}\right) = \frac{2\pi}{3} \text{ rad}$$

b) $\dfrac{\pi}{3}$ radian to degree.

Solution:

$$\frac{\pi}{3}\text{ rad} = \frac{\pi}{3}\left(\frac{180}{\pi}\text{ deg}\right) = 60\text{ deg}$$

10.2 ANGULAR MOTION VARIABLES

Angular position (θ) of a particle is defined to be the angle formed between its position vector and the positive x-axis. Unit of measurement for angular position is radian. Angular position is a vector whose direction is perpendicular to the plane determined by the angle.

Angular displacement $(\Delta\theta)$ is defined to be the change in the angular position of a particle. Angular displacement is a vector quantity whose direction is perpendicular to the plane of the angular displacement.

$$\Delta\theta = \theta_f - \theta_i$$

Average Angular velocity $(\bar{\omega})$ is defined to be angular displacement per a unit time. Unit of measurement is rad/s. Angular velocity is a vector quantity whose direction is along the axis of rotation.

$$\bar{\omega} = \frac{\Delta\theta}{\Delta t} = \frac{\theta_f - \theta_i}{\Delta t}$$

Instantaneous Angular Velocity (ω) is angular velocity at a given instant of time.

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Average angular acceleration $(\bar{\alpha})$ is change in angular velocity per a unit time. Its unit of measurement is radians/second^2. It is a vector quantity whose direction is along the direction of change in angular velocity.

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{\omega_f - \omega_i}{\Delta t}$$

Instantaneous angular acceleration (α) is angular acceleration at a given instant of time.

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt}$$

10.3 RELATIONSHIP BETWEEN LINEAR AND ANGULAR VARIABLES

Distance or arc-length *s*, angular position θ and radius *r* are related by $s = r\theta$. Taking the changes of both sides of the equations, a relationship between linear displacement and angular displacement is obtained.

$$\Delta s = r\Delta\theta$$

Dividing both sides of this equation by the interval of time Δt during which these displacements took place, the equation $\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$ is obtained. But $\frac{\Delta s}{\Delta t}$ is equal to the linear speed v and $\frac{\Delta\theta}{\Delta t}$ is equal to the angular speed ω.

$$v = r\omega$$

Taking the change of both sides of this equation and then dividing by Δt the equation $\frac{\Delta v}{\Delta t} = r\frac{\Delta\omega}{\Delta t}$ is obtained. But $\frac{\Delta v}{\Delta t}$ is equal to the tangential acceleration $a_\theta = a_t$ and $\frac{\Delta\omega}{\Delta t}$ is equal to the angular acceleration α.

$$a_t = r\alpha$$

10.4 UNIFORMLY ACCELERATED ANGULAR MOTION

Uniformly accelerated angular motion is motion with constant angular acceleration. The equations for a uniformly accelerated motion are obtained in the same way as the equations for a uniformly accelerated linear motion. Hence, they can easily be obtained from the equations of a uniformly accelerated motions by replacing linear variables with angular variables $\left(\text{that is } \Delta x \to \Delta\theta;\ v \to \omega;\ a \to \alpha\right)$.

$$\omega_f = \omega_i + \alpha t$$

$$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 = w_i^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \left(\frac{\omega_i + \omega_f}{2}\right)t$$

Only two of these equations are independent. If any 3 of the 5 variables are known, the other 2 can be obtained by using these equations.

Example: The angular speed of a rigid object rotating about a fixed axis increased from 10 rad/s to 30 rad/s in 10 seconds.

a) Calculate its angular acceleration.
Solution:
$\omega_i = 10 \text{ rad/s};\ \omega_f = 30 \text{ rad/s};\ t = 10 \text{ s};\ \alpha = ?$

$$\omega_f = \omega_i + \alpha t$$

$$30 \text{ rad/s} = 20 \text{ rad/s} + \alpha(10 \text{ s})$$

$$\alpha = 2 \text{ rad/s}^2$$

b) Calculate its angular displacement.
Solution:
$\Delta\theta = ?$

$$\Delta\theta = \omega_i t + \frac{1}{2}\alpha t^2$$

$$\Delta\theta = \left((10)(10) + \frac{1}{2}(2)(10)^2\right) \text{ rad} = 200 \text{ rad}$$

c) For a particle on the rigid object at a perpendicular distance of 10 cm from the axis of rotation, calculate

a) Its tangential acceleration.

Solution:

$r = 0.1$ m; $a_t = ?$

$$a_t = r\alpha = (0.1)(2)\ \text{m/s}^2 = 0.2\ \text{m/s}^2$$

b) the distance travelled.

Solution:

$\Delta s = ?$

$$\Delta \text{s} = \text{r}\Delta\theta = (0.1)(200)\ \text{m} = 20\ \text{m}$$

c) Its final linear speed.

$v_f = ?$

$$v_f = r\omega_f = (0.1)(30)\ \text{m/s} = 3\ \text{m/s}$$

10.5 MOMENT OF INERTIA

Moment of inertia is to rotational motion as mass is to linear motion. The moment of inertia (I) of a particle of mass m located at a perpendicular distance $r_\perp$ from the axis of rotation is defined as

$$I = mr_\perp^2$$

The unit of measurement for moment of inertia is kg m^2. The moment of inertia of system of particles is obtained by adding the moment of inertia of all of the particles.

$$I = \sum_i m_i r_{\perp i}^2 = m_1 r_{\perp 1}^2 + m_2 r_{\perp 2}^2 + \dots$$

10.6 ROTATIONAL KINETIC ENERGY

Suppose a particle of mass m is revolving about a fixed axis at a perpendicular distance of $r_\perp$ from the axis with a speed of v Its speed is related to its angular speed by $v = \omega r_\perp$. Its kinetic energy is $KE = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega r_\perp)^2 = \frac{1}{2}(mr_\perp{}^2)\omega^2$. But $mr_\perp{}^2$ is the moment of inertia of the particle about the axis of rotation. Therefore the rotational kinetic energy (KE_{rot}) of the particle is given as $KE_{rot} = \frac{1}{2}I\omega^2$. If there are a number of particles, the total kinetic energy is the sum of the kinetic energies of the particle. Since all of the particles are moving with the same angular speed, $KE_{rot} = \frac{1}{2}\sum_i I_i\omega^2 = \frac{1}{2}\omega^2\sum_i I_i$. But $\sum_i I_i = I$ is the moment of inertial of the system of particles. Therefore for any system of particles rotating about a certain axis of rotation with angular speed ω, the rotational kinetic energy is given as

$$KE_{rot} = \frac{1}{2}I\omega^2$$

Provided I is the moment of inertia of the particles about the axis of rotation.

Example: Four particles of masses 1 kg, 2 kg, 3 kg and 4 kg are located at the points (0, 4) m, (2, –3) m, (4, 1) m and (–4, 5) m respectively. If this system of particles is revolving around the y-axis with an angular speed of 2 rad/s,

a) Calculate the moment of inertia of the system.
Solution: For particles revolving around the x-axis, the perpendicular distance between a particle and the axis of rotation is equal to the absolute value of the x-coordinate of the particle.
$m_1 = 1$ kg; $r_{\perp 1} = |x_1| = 0$; $m_2 = 2$ kg; $r_{\perp 2} = |x_2| = 2$ m; $m_3 = 3$ kg; $r_{\perp 1} = |x_3| = 4$ m;

$m_4 = 4$ kg; $r_{\perp 4} = |x_4| = 4$ m; $I = ?$

$$I = \frac{1}{2}m_1 r_{\perp 1}{}^2 + \frac{1}{2}m_2 r_{\perp 2}{}^2 + \frac{1}{2}m_3 r_{\perp 3}{}^2 + \frac{1}{2}m_4 r_{\perp 4}{}^2$$
$$= \left[\frac{1}{2}(1)(0)^2 + \frac{1}{2}(2)(2)^2 + \frac{1}{2}(3)(4)^2 + \frac{1}{2}(4)(4)^2\right] \text{ kg m}^2 = 60 \text{ kg m}^2$$

b) Calculate the rotational kinetic energy of the system.

Solution:

$\omega = 2$ rad/s; $KE_{rot} = ?$

$$KE_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}\times 60 \times 2^2 \text{ J} = 120 \text{ J}$$

Practice Quiz 10.1

Choose the best answer

1. Convert *6.2* radians to degrees.
   A) *136.642*3l deg
   B) *620.886*3l deg
   C) *355.234*3l deg
   D) *531.515*3l deg
   E) *487.872*3l deg

2. Convert *351* degree to revolution.
   A) *0.975* rev
   B) *1.387* rev
   C) *0.764* rev
   D) *0.509* rev
   E) *0.356* rev

3. Calculate the length of an arc subtended (opened) by a central angle whose radian measure is *4.5* rad in a circle of radius *2.33* .44m.
   A) *3.069* m
   B) *12.014* m
   C) *10.485* m
   D) *16.886* m
   E) *8.966* m

4. A wheel of radius *14.33* m was rolled for a distance of *28* m. Calculate the angular displacement of a point on the rim of the wheel.
   A) *3.655* rad
   B) *1.132* rad
   C) *3.151* rad
   D) *2.595* rad
   E) *1.954* rad

5. The angular speed of an object rotating about a fixed axis changed from *12* rad / s to *15* rad / s uniformly in *0.53* seconds. Calculate its angular acceleration.
   A) *2.987* rad/s^2
   B) *5.66* rad/s^2
   C) *3.847* rad/s^2
   D) *10.088* rad/s^2
   E) *5.018* rad/s^2

6. The angular speed of an object rotating about a fixed axis changed from *12.9* rad/s to *13* rad/s uniformly in *2.21* seconds. Calculate its angular displacement.
   A) *4.345* rad
   B) *28.62* rad
   C) *36.838* rad
   D) *50.162* rad
   E) *33.38* rad

7. Starting with an angular speed of *16* rad / s, an object was accelerated uniformly about a fixed axis with an acceleration of *6* rad / s^2 for *27.77* s. Calculate its angular displacement.
   A) *2757.839* rad
   B) *4946.123* rad
   C) *1011.214* rad
   D) *1810.592* rad
   E) *4311.533* rad

8. Starting with an angular speed of *18* rad / s, an object was accelerated uniformly about a fixed axis with an acceleration of *2* rad / s^2 for *15.74* s. Calculate the final linear speed of a particle located at a perpendicular distance of *1.2* from the axis of rotation.
   A) *24.994* m / s
   B) *97.496* m / s
   C) *33.999* m / s
   D) *59.376* m / s
   E) *82.359* m / s

9. Moment of inertia of a particle is
   A) Proportional to the square of the perpendicular distance between the axis of rotation and the particle.
   B) Proportional to the cube of the perpendicular distance between the axis of rotation and the particle.
   C) proportional to the perpendicular distance between the axis of rotation and the particle
   D) Inversely proportional to the perpendicular distance between the axis of rotation and the particle.
   E) Inversely proportional to the square of the perpendicular distance between the axis of rotation and the particle.

10. Three particles of masses *10.3* kg, *6.5* kg and *4.7* kg are located at the points (*3.4, -1.1*) m, (*-11.3, 1.6*) m, and (*8, 2*) m respectively. Calculate the moment of inertia of these system of particles if the axis of rotation is the x-axis.
    A) *24.291* kg m ^{2}
    B) *69.78* kg m ^{2}
    C) *10.703* kg m ^{2}
    D) *47.903* kg m ^{2}
    E) *89.692* kg m ^{2}

11. Three particles of masses *8.3* kg, *9.5* kg and *1.7* kg are located at the points (*-6.8, 1.7*) m, (*4, 1*) m, and (*5, -5*) m respectively. Calculate the rotational kinetic energy of this system of particles if they are revolving around the x-axis with an angular velocity of *12* rad / s.
    A) *7820.208* J
    B) *9742.852* J
    C) *5471.064* J
    D) *2663.796* J
    E) *3760.195* J

10.7 MOMENT OF INERTIA OF SOLID OBJECTS

The moment of inertia of a solid object may be obtained by treating the solid object as made up of small mass elements, Δm_i's, and then taking the limiting value as the Δm_i's approach zero which of course makes it an integral. That is $I = \lim_{\Delta m_i \to 0} \sum_i \Delta m_i \, r_{\perp i}^2$ or

$$I = \int r_{\perp}^2 dm$$

Where $r_{\perp}$ is the perpendicular distance between the small mass element *dm* and the axis of rotation.

Example: Obtain the moment of inertia of a uniform thin rod of length, *L*, and mass, *M*, about an axis passing through the midpoint of the rod perpendicularly.

Solution: Let's use a coordinate system where the x-axis lies along the rod and the y-axis passes through the midpoint of the rod perpendicularly. In other words the y-axis is the axis of rotation. Let the thickness and the x-coordinate of the arbitrary mass element dm be dmx and x respectively. The value of x varies from $-\frac{L}{2}$ to $\frac{L}{2}$. Since the rod is uniform $\frac{dm}{dx}=\frac{M}{L}$ or $dm=\frac{M}{L}dx$. Therefore $r_\perp{}^2 dm = x^2\frac{M}{L}dx$ and the moment of inertia can be obtained as $I=\int r_\perp{}^2 dm = \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\frac{M}{L}dx = \frac{ML^2}{12}$

Example: Find the moment of inertia of a uniform rod od length L, and mass M about an axis that passes through one of its end perpendicularly.

Solution: Let's use a coordinate system where the x-axis lies along the rod and the y-axis passes through the left end perpendicularly. With this coordinate system, the y-axis is the axis of rotation. Consider a small path element of size *dx* and mass *dm* whose x-coordinate is *x*. The value of *x* varies from 0 to *L* along the rod. Since the rod is uniform, $\frac{dm}{dx}=\frac{M}{L}$ or $dm=\frac{M}{L}dx$. Therefore $r_\perp{}^2dm=x^2\frac{M}{L}dx$ and the moment of inertia can evaluated as $I=\int r_\perp{}^2dm=\int_0^L x^2\frac{M}{L}dx=\frac{ML^2}{3}$.

Example: Obtain the moment of inertia of a uniform thin ring of mass, *M*, and radius, *R*, about an axis that passes through its center perpendicularly.

Solution: Any part of the ring has the same perpendicular distance from ring which is the radius. That is $r_\perp=R$ and $I=\int r_\perp{}^2dm=\int R^2dm=R^2\int dm=MR^2$.

Example: Obtain the moment of inertia of a uniform disc of mass *M* and radius *R* about an axis that passes through its center perpendicularly.

Solution: Let's take the small mass element, *dm*, to be the mass of a thin ring of radius *r* and thickness *dr*. Its moment of inertia is r^2dm. The total moment of inertia of the disc is obtained by adding the moment of inertial of all the rings or by integrating from 0 to *R*. $I=\int_0^R r^2dm$. Since it is uniform $\frac{dm}{dA}=\frac{M}{\pi R^2}$ or $dm=\frac{M}{\pi R^2}dA$ where *dA* is the area of the ring which is equal to $2\pi rdr$. Thus $dm=\frac{2M}{R^2}rdr$. Now the moment of inertia of the disc can be obtained as $I=\int_0^R r^3\left(\frac{2M}{R^2}\right)dr=\frac{MR^2}{2}$.

Example: Obtain the moment of inertia of a uniform solid cylinder of mass, *M*, radius *R* and length, *L*, about an axis that passes through its axis.

Solution: Let's use a coordinate system where the z-axis lies along the axis of the cylinder and the base of the cylinder lies on the xy-plane. Let's take the small mass element, *dm*, to be the mass of a disc of radius *R* and thickness *dz* at elevation of *z* The value of *z* varies from 0 to *L*. The moment of inertia of this disc is $\frac{R^2dm}{2}$. The moment of inertia of the cylinder is obtained $I=\int_0^M\frac{R^2dm}{2}=\frac{MR^2}{2}$.

Example: The linear density of a rod that extends from the origin to *x* = 2 m along the x-axis depends on x according to the equation $\lambda=ax+b$ where $a=1$ kg/m^2 and $b=4$ kg/m. Calculate the moment of inertia of the rod about a vertical axis along the $x=4$ m line.

Solution: Let the thickness and the coordinate of an arbitrary mass element *dm* be *dx* and *x* respectively. Then $r_\perp = 4 - x$ and $\lambda = \frac{dm}{dx} = ax + b$ or $dm = (ax+b)dx$. Therefore the moment of inertia of the rod is given as $I = \int r_\perp^2 dm = \int_0^2 (4-x)^2 (ax+b)dx = 89.3 \text{ kg m}^2$.

Example: The density of a thin ring of radius 0.2 m depends on angle θ (direction of position vector) according to the equation $\lambda = \sqrt{\theta} + 1$. Calculate the moment of inertia about an axis that passes through the center of the ring perpendicularly.

Solution: Let the size of an arbitrary mass element *dm* on the ring be *ds*. Then $\frac{dm}{ds} = \lambda = \sqrt{\theta} + 1$ or $dm = \left(\sqrt{\theta}+1\right)ds$ And with $ds = rd\theta = 0.2d\theta$, $dm = 0.2\left(\sqrt{\theta}+1\right)d\theta$. All of the mass elements are at the same distance from the axis of rotation: $r_\perp = 0.2 \text{ m}$. Therefore the moment of inertia is given as $I = \int r_\perp^2 dm = \int_0^{2\pi} 0.2^2 \left(0.2\left(\sqrt{\theta}+1\right)d\theta\right) = 0.134 \text{ kg m}^2$.

10.8 THE PARALLEL AXIS THEOREM

The *parallel axis theorem* is a theorem that relates the moment of inertia about a certain axis of rotation with the moment of inertial about a parallel axis of rotation that passes through the center of mass of the object. Formulae for the moment of inertias about an axis that passes through the center of mass are available in textbooks for different shapes. The parallel axis theorem is useful in obtaining moment of inertia about an axis that doesn't pass through the center of mass.

For simplicity, let's consider a rod with a coordinated system where the x-axis lies along the rod and the y-axis passes through the center of mass perpendicularly. Let the axis of rotation through the center of mass be the y-axis itself. Let the moment of inertia about this axis be denoted as I_{CM} Let the coordinate of an arbitrary mass element *dm* be *x*. The perpendicular distance between *dm* and the axis through the center of mass is $|x|$ and thus $I_{CM} = \int x^2 dm$. Now consider another axis parallel to this axis that lies along the line $x = d$. Let the moment of inertia about this axis be denoted by *I*. The perpendicular distance between *dm* and this axis is $|x-d|$ and therefore $I = \int r_\perp^2 dm = \int (x-d)^2 dm = \int x^2 dm - 2d\int xdm + d^2 \int dm$. But, $\int dm = M$, $\int x^2 dm = I_{CM}$ and $\int xdm = Mx_{CM}$. $x_{CM} = 0$ because the center of mass is located at the origin. Therefore I and I_{CM} are related by the following equation:

$$I = I_{CM} + Md^2$$

This is a mathematical statement of the parallel axis theorem. The *parallel axis theorem* states that if the moment of inertia of an object of mass M about an axis that passes through its center of mass is I_{CM}, then the moment of inertia (I) about another parallel axis at a perpendicular distance of d from this axis is equal to the sum of I_{CM} and the product of the mass of the object and the square of d.

Example: The moment of inertia of a uniform thin rod of mass M and length L about an axis that passes through its center of mass perpendicularly is given by $I_{CM} = \frac{ML^2}{12}$. Use the parallel axis theorem to show that its moment of inertia about a parallel axis through one of its end points is given by $\frac{ML^2}{3}$.

Solution: The perpendicular distance the axis through the center of mass and the parallel axis through the end point half of the length of the rod.

$$d = \frac{L}{2};\ I = ?$$

$$I = I_{CM} + Md^2 = \frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}$$

10.9 ROLLING MOTION

A rolling object has two kinds of kinetic energy. It has linear kinetic energy because it is being displaced linearly. It also has rotational kinetic energy because it rotates as it is displaced linearly. Let the position vector of a small mass element dm be $\vec{r}$. If $\vec{r}_{CM}$ is the position vector of the center of mass and $\vec{r}\,'$ is the position vector of the small mass element dm with respect to the center of mass, then $\vec{r} = \vec{r}_{CM} + \vec{r}\,'$ which implies $\frac{d\vec{r}}{dt} = \frac{d\vec{r}_{CM}}{dt} + \frac{d\vec{r}\,'}{dt}$. The kinetic energy of the small mass element dm is given as $dKE = \frac{1}{2}dm\left(\frac{d\vec{r}}{dt}\right)^2$ and thus the total kinetic energy may be expanded as $KE = \frac{1}{2}\int\left(\frac{d\vec{r}}{dt}\right)^2 dm = \int\left(\frac{d\vec{r}_{CM}}{dt} + \frac{d\vec{r}\,'}{dt}\right)^2 dm = \frac{1}{2}\int\left(\frac{d\vec{r}_{CM}}{dt}\right)^2 dm + \int\frac{d\vec{r}_{CM}}{dt}\cdot\frac{d\vec{r}\,'}{dt}dm + \frac{1}{2}\int\left(\frac{d\vec{r}\,'}{dt}\right)^2 dm$

$\frac{d\vec{r}_{CM}}{dt} = \vec{v}_{CM}$ is the velocity of the center of mass and is independent of the mass elements. Therefore $\frac{1}{2}\int\left(\frac{d\vec{r}_{CM}}{dt}\right)^2 dm = \frac{1}{2}\vec{v}_{CM}{}^2\int dm = \frac{1}{2}M\vec{v}_{CM}{}^2$.

$\int\frac{d\vec{r}_{CM}}{dt}\cdot\frac{d\vec{r}\,'}{dt}dm = \vec{v}_{CM}\cdot\int\frac{d\vec{r}\,'}{dt}dm.$ But $\int\frac{d\vec{r}\,'}{dt}dm = M\vec{v}\,'_{CM}$ where $\vec{v}\,'_{CM}$ is the velocity of the center mass with respect to a coordinate system (primed coordinate system) whose origin is fixed at the center of mass itself which should be zero of course. Therefore $\int\frac{d\vec{r}_{CM}}{dt}.\frac{d\vec{r}\,'}{dt}dm = 0.$

$\frac{d\vec{r}\,'}{dt}$ is the velocity of the mass element dm with respect to the center of mass. Since the distance between the center of mass and is cannot change, this velocity can only be rotational velocity about the center of mass. If the angular speed is denoted by ω then $\left|\frac{d\vec{r}\,'}{dt}\right| = r'\omega.$ Therefore $\frac{1}{2}\int\left(\frac{d\vec{r}\,'}{dt}\right)^2 dm = \frac{1}{2}\int(r'\omega)^2 dm = \frac{\omega^2}{2}\int(r')^2 dm$ (All of the particles share the same angular speed). Since r' is the perpendicular distance between the center of mass and dm, $\int(r')^2 dm = I_{CM}$.

Therefore the total energy of a rolling object is given as follows.

$$KE = \frac{1}{2}Mv_{CM}{}^2 + \frac{1}{2}I_{CM}\omega^2$$

v_{CM} is the linear speed of the object. ω is angular velocity about an axis through the center of mass. The point of contact between the rolling object and the surface travels with the same velocity as the object, but also the velocity of the point of contact is equal to $R\omega$ where R is the distance between the center of mass and the point of contact.

$$v_{CM} = R\omega$$

Example: A cylinder of radius 4 cm and mass 2 kg is rolling on a horizontal surface with a velocity of 0.2 m/s. Calculate its kinetic energy. For a cylinder $I_{CM} = \frac{MR^2}{2}$.

Solution:
$M = 2$ kg; $R = 0.04$ m; $v_{CM} = 0.2$ m/s; $KE = ?$

$$\omega = \frac{v_{CM}}{R} = \frac{0.2}{.04} \text{ rad/s} = 5 \text{ rad/s}$$

$$I_{CM} = \frac{MR^2}{2} = \frac{2\times 0.04^2}{2} \text{ kg m}^2 == 1.6\times 10^{-3} \text{ kg m}^2$$

$$KE = \frac{1}{2}Mv_{CM}{}^2 + \frac{1}{2}I_{CM}\omega^2 = \frac{1}{2}(2)(0.2)^2 \text{ J} + \frac{1}{2}(1.6\times 10^{-3}(5)^2 \text{ J} = 0.04 \text{ J}$$

Example: A sphere is rolling down a 10 m, $37°$ inclined plane. Calculate its speed at the bottom $\left(I_{sphere} = \frac{2}{5}MR^2\right)$.

Solution: The forces acting on a rolling sphere are gravity and friction. Gravity is conservative but friction is not. But in this case the effect of friction is to produce rotational energy of the object not to lose energy. Therefore if rotational energy is included the principle of conservation of momentum applies.

$v_i = 0$; $d = 10$ m; $\theta = 37°$; $v_f = 0$

$$\omega_i = \frac{v_i}{R} = 0$$

$$y_i - y_f = 10\sin(37°) \text{ m} = 6 \text{ m}$$

$$KE_i + U_{gi} = KE_f + U_{gf}$$

$$\frac{1}{2}mv_i^2 + \frac{1}{2}I_{CM}\omega_i^2 + m|g|y_i = \frac{1}{2}mv_f^2 + \frac{1}{2}I_{CM}\omega_f^2 + m|g|y_f$$

$$m|g|(y_i - y_f) = \frac{1}{2}mv_f^2 + \frac{1}{2}I_{CM}\omega_f^2$$

$$m|g|(y_i - y_f) = \frac{1}{2}mv_f^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v_f}{R}\right)^2$$

$$m|g|(y_i - y_f) = \frac{1}{2}mv_f^2 + \frac{1}{5}mv_f^2 = \frac{7}{10}mv_f^2$$

$$v_f = \sqrt{\frac{10}{7}|g|(y_i - y_f)} = \sqrt{\frac{10}{7}(10)(6)} \text{ m/s} = 9.25 \text{ m/s}$$

Practice Quiz 10.2

Choose the best answer

1. Calculate the moment of inertia of a uniform spherical object of radius *0.24* m and density *1933* kg / m^3 about any axis that passes through its center. ($I_{CM} = 2MR^2/5$)
   A) *0.564* kg m^2
   B) *2.579* kg m^2
   C) *3.745* kg m^2
   D) *1.735* kg m^2
   E) *2.131* kg m^2

2. Calculate the rotational kinetic energy of a uniform cylindrical object of radius *0.34* m, height *0.2* m and density *1933* kg / m^3 if it is rotating about an axis that passes through its axis with an angular velocity of *6.9* rad / s. ($I_{CM} = MR^2/2$)
   A) *272.305* J
   B) *142.903* J
   C) *312.994* J
   D) *193.182* J
   E) *243.532* J

3. The linear density of a rod that extends from $x = 1$ m to $x = 3$ m varies with x according to the formula $\lambda = x^{0.3}$ Calculate it's moment of inertia about a vertical axis along the line $x = 5$ m.
   A) 25.739 kg m^2
   B) 3.428 kg m^2
   C) 35.932 kg m^2
   D) 21.998 kg m^2
   E) 6.399 kg m^2

4. The linear density of a ring of radius 0.3 m centered at the origin varies on θ (direction of position vector) according to the equation $\lambda = \theta^{0.2} + 1$. Calculate the moment of inertia of the ring about an axis that passes through its center and is perpendicular to the plane of the ring.
   A) 0.568 kg m^2
   B) 0.374 kg m^2
   C) 0.529 kg m^2
   D) 0.318 kg m^2
   E) 0.466 kg m^2

5. Calculate the moment of inertia of a uniform rod of mass 0.6 kg and length 2 m about an axis that passes through the rod perpendicularly at a point 1.3 m away from its left end.($I_{CM} = ML^2 / 12$)
   A) 0.197 kg m^2
   B) 0.323 kg m^2
   C) 0.254 kg m^2
   D) 0.406 kg m^2
   E) 0.169 kg m^2

6. Calculate the kinetic energy of a solid disc of radius 0.7 m and mass 2.5 kg which is rotating about an axis that is tangent to the disc perpendicularly with an angular speed of 5 rad / s. ($I_{CM} = MR^2 / 2$)
   A) 10.734 J
   B) 36.998 J
   C) 34.139 J
   D) 22.969 J
   E) 29.716 J

7. The centers of two identical spheres (connected by a rod of negligible mass) of radius *0.8* m and mass *7.2* kg are located on the x-axis at $x = -3$ m and $x = 2$ m. Calculate the moment of inertia about the vertical axis along $x = 0$. ($I_{CM} = 2MR^2/5$)
   A) *156.499* kg m^2
   B) *133.79* kg m^2
   C) *116.33* kg m^2
   D) *82.245* kg m^2
   E) *97.286* kg m^2

8. A cylindrical object of mass *0.5* kg and radius *0.016* m is rolling on a horizontal surface with a speed of *5* m/s. Calculate its kinetic energy. ($I_{\text{cylinder}} = MR^2/2$)
   A) *11.933* J
   B) *9.375* J
   C) *13.263* J
   D) *6.151* J
   E) *3.441* J

9. A spherical object of mass *2* kg and radius *0.022* m is rolling down an inclined plane of length *3* m that makes an angle of *70* deg with the ground. Calculate its speed by the time it reaches the ground ($I_{\text{spehre}} = 2MR^2/5$)
   A) *6.282* m/s
   B) *9.962* m/s
   C) *5.399* m/s
   D) *10.641* m/s
   E) *8.453* m/s

10. A spherical object of mass *5* kg and radius *0.016* m is rolled upwards from the bottom of an inclined plane of inclination *10* deg with a speed of *20.6* m/s. How high would the object rise? ($I_{\text{spehre}} = 2MR^2/5$)
   A) *23.755* m
   B) *39.337* m
   C) *36.035* m
   D) *11.629* m
   E) *30.311* m

11 TORQUE AND ANGULAR MOMENTUM

Your goal for this chapter is to learn about torque, relationships between torque and rotational motion variables, and angular momentum.

Torque $(\vec{\tau})$ is a vector physical quantity used as a measure of the rotational effect of force. Its magnitude (τ) is proportional to the magnitude of the force and to the perpendicular distance between the point of rotation and the line of action of the force.

$$\tau = Fr_{\perp}$$

Where τ and F represent the magnitude of the torque and force respectively; and $r_\perp$ represents the perpendicular distance between the point of rotation and the line of action of the force. If r is the distance between the point of rotation and the point of application of the force and θ is the angle between the position vector of the point of application of force with respect to the point of rotation (The vector whose tail is at the point of rotation and whose head is at the point of application of force), the $r_\perp = r\sin(\theta)$. Thus

$$\tau = Fr\sin(\theta)$$

The direction of torque is perpendicular to the plane determined by the force vector and the position vector of the point of application of force (with respect to the point of rotation). It is perpendicularly out if the tendency of the force is to produce counterclockwise rotation and perpendicularly in if the tendency of the force is to produce clockwise rotation. The component of torque (τ) is taken to be positive if the tendency of the force is to produce counterclockwise rotation and negative if the tendency of the force is to produce clockwise rotation. The plane of rotation can be taken to be the xy-plane without any loss of generality. Then the direction of torque will be along the z-axis. That is, $\vec{\tau} = \tau_z\hat{k}$ where τ_z is the z-component of the torque. τ_z is positive (negative) if the tendency of the torque is to produce counterclockwise (clockwise) rotation and can be written in terms of the magnitude of torque as

$$\tau_z = \pm\tau = \pm Fr\sin(\theta)$$

The unit of measurement for torque is Newton meter (N m).

Example: A horizontal lever of length 4 m is pivoted at its center. A vertically downward force of 5 N is applied at a distance of 0.5 m to the right of the pivot. Calculate the torque acting on the lever.

Solution: The force has the tendency of producing clockwise rotation. Therefore the torque is negative.
$F = 5$ N; $r = 0.5$ m; $\theta = 90°$; $\tau_z = ?$

$$\tau_z = -Fr\sin(\theta) = -5\times 0.5\times\sin(90°)\text{ N m} = -2.5\text{ N m}$$

Example: A horizontal lever of length 6 m is pivoted at its center. A force of 10 N with an upward vertical component makes an angle of 30° with the positive x-axis is acting at the right end of the lever. Calculate the torque acting on it.

Solution: The torque is positive because the tendency of the force is to produce counterclockwise rotation.
$F = 10$ N; $r = 3$ m; $\theta = 30°$; $\tau_z = ?$

$$\tau_z = Fr\sin(\theta) = 10 \times 3 \times \sin(30°) \text{ N m} = 15 \text{ N m}$$

Example: A horizontal lever of length 3 m is pivoted at its center. A horizontal force of 40 N ia pulling to the left at its left end. Calculate the torque acting on it.

Solution: This force does not have any rotational effect on the lever. The torque is expected to be zero.
$F = 40$ N; $r = 1.5$ m; $\theta = 0°$; $\tau_z = ?$

$$\tau_z = Fr\sin(\theta) = 40 \times 1.5 \times \sin(0°) = 0$$

11.1 NET TORQUE

Net torque acting on an object is the vector sum of all the torques acting on the object.

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \dots$$

If the object is rotating in a plane, all the torques have the same line of action (either perpendicularly out or perpendicularly in) and this vector equation can be described by a single component equation.

$$\tau_z = \tau_{1z} + \tau_{2z} + \tau_{3z} + \dots$$

Example: A horizontal uniform lever of length 4 m is pivoted at its center. A force of 5 N is pulling vertically down ward at the right end of the lever. A force of 10 N that makes an angle of 37° with the negative x-axis is pulling the lever upward at its right end. A force of 2 N is pushing vertically downward at a distance of *1* m to the left of the pivot. Calculate the net torque acting on the lever and determine whether it is rotating clockwise or counterclockwise.

Solution: The torque due to the *5* N force is negative because it has a tendency of causing clockwise rotation. The torque due to the *10* N force is positive because its tendency is to cause counterclockwise rotation. The torque due to the *2* N force is positive because it has a tendency of causing counterclockwise rotation.

Solution:

$F_1 = 5$ N; $\theta_1 = 90°$; $r_1 = 2$ m; $F_2 = 10$ N; $\theta_2 = 37°$; $r_2 = 2$ m; $F_3 = 2$ N; $\theta_3 = 90°$; $r_3 = 1$ m; $\tau_z = ?$

$$\tau_{1z} = F_1 r_1 \sin(\theta_1) = -5 \times 2 \times \sin(90°) \text{ N m} = -10 \text{ N m}$$

$$\tau_{2z} = F_2 r_2 \sin(\theta_2) = 10 \times 2 \sin(37°) \text{ N m} = 12 \text{ N m}$$

$$\tau_{3z} = F_3 r_3 \sin(\theta_3) = 2 \times 1 \sin(90°) \text{ N m} = 2 \text{ N m}$$

$$\tau_z = \tau_{1z} + \tau_{2z} + \tau_{3z} = (-10 + 12 + 2) \text{ N m} = 4 \text{ N m}$$

It is rotating counterclockwise because the net torque is positive.

11.2 TORQUE AS A CROSS PRODUCT

Torque about a point: Torque about a point is a torque where rotation about any axis is possible. If $\vec{r}$ is the position vector of the point of application of force with respect to the point of rotation, then torque is equal to the cross product between the position vector and the force.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The magnitude of the torque is given as $\tau = Fr\sin(\theta)$ where F stands for the magnitude of the force, r stands for the distance between the point of rotation and the point of application of force and θ is the angle formed between the force and the position vector of the point of application of force with respect to the point of rotation. The direction of torque is perpendicular to the plane determined by $\vec{r}$ and $\vec{F}$. To distinguish between the two possible directions (perpendicularly out of the plane denoted by a dot $(\cdot)$ and perpendicularly into the plane denoted by a cross $(\times)$), the screw rule or the right hand rule discussed on the chapter of vectors can be used. If the position vector and the force are known in terms of their Cartesian components ($\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{F} = F_x\hat{i} + F_y\hat{j} + F_y\hat{k}$), then the cross product can be simplified using the cross products between the unit vectors ($\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$, $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{i} = -\hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{j} = -\hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$, $\hat{i}\times\hat{k} = -\hat{j}$) to give the following expression for the cross product in terms of Cartesian coordinates.

$$\vec{\tau} = \left(yF_z - zF_y\right)\hat{i} + \left(zF_x - xF_z\right)\hat{j} + \left(xF_y - yF_x\right)\hat{k}$$

Torque about a fixed axis: Torque about a fixed axis is a torque where only rotation about one direction is possible. Using a coordinate system where the z-axis lies along the axis of rotation, the torque can have a z-component. The x-component and y-component of the torque are not possible. Therefore $\vec{\tau} = \tau_z\hat{k} = \left(xF_y - yF_x\right)\hat{k}$ which is equal to the cross product between the projection of the position vector on the xy-plane $\left(\vec{r}_\perp = x\hat{i} + y\hat{j}\right)$ and the projection of the force on the xy-plane $\left(F_\perp = F_x\hat{i} + F_y\hat{j}\right)$. If $\hat{e}_{r_\perp}$ is a radially outward unit vector on the xy-plane and $\hat{e}_\theta$ is a unit vector tangent to a circle on the xy-plane concentric the z-axis in a counterclockwise direction, then $\vec{r}_\perp = r_\perp\hat{e}_{r_\perp}$ and $\vec{F}_\perp = F_{r_\perp}\hat{e}_{r_\perp} + F_\theta\hat{e}_\theta$. Thus $\vec{\tau} = r_\perp e_{r_\perp} \times \left(F_{r_\perp}\hat{e}_{r_\perp} + F_\theta\hat{e}_\theta\right) = r_\perp F_\theta\left(\hat{e}_{r_\perp}\times\hat{e}_\theta\right)$ or

$$\vec{\tau} = r_\perp F_\theta\hat{k}$$

Where $r_\perp$ is the perpendicular distance between the axis of rotation and the point of application of force; and F_θ is the component of the forces in the direction the unit vector tangent to a circle concentric to the z-axis in a counterclockwise rotation. F_θ is positive if its direction is in a counterclockwise and negative if its direction is clockwise. That is the direction of the torque is along the positive z-axis $\left(\hat{k}\right)$ if the tendency of the torque is to produce counterclockwise rotation; and is along the negative z-axis $\left(-\hat{k}\right)$ if the tendency of the force is to produce clockwise rotation. In other words, the direction of torque and the direction of rotation are related by the right hand rule. When fingers are wrapped in the direction of rotation, then thumb points in the direction of rotation.

Example: A rod is pivoted freely at one of its end points. If a force of $\left(3\hat{i} + 2\hat{j} + 4\hat{k}\right)$ is acting at the other end of the rod by the time this end of the rod is located at the point $\left(-2i + j + 5\hat{k}\right)$ with respect to the other end, calculate the torque acting on the rod.

Solution:

$\vec{\tau} = ?$

$$\vec{\tau} = \vec{r} \times \vec{F} = \left(-2i + j + 5\hat{k}\right) \times \left(3\hat{i} + 2\hat{j} + 4\hat{k}\right) \text{ N m}$$

$$= \left(\left(1\times 4 - 5\times 2\right)\hat{i} + \left(5\times 3 + 2\times 4\right)\hat{j} + \left(-2\times 2 - 1\times 3\right)\hat{k}\right) \text{ N m}$$

$$= \left(-6\hat{i} + 23\hat{j} - 7\hat{k}\right) \text{ N m}$$

Example: The position vector of a particle varies according to the equation $\vec{r} = at^3\hat{i} + bt\hat{j} + c\hat{k}$ where $a = 2\text{m/s}^3$, $b = 4$ m/s; $c = 8$ with respect to the point of rotation. If the mass of the particle is 2 kg, calculate the torque acting on it after 2 seconds.

Solution:

$\vec{\tau}\big|_{t=2\text{ s}} = ?$

$$\vec{r}\big|_{t=2\text{ s}} = \left(16\hat{i} + 8\hat{j} + 8\hat{k}\right) \text{ m}$$

$$F(t) = m\frac{d^2\vec{r}}{dt^2} = (2\text{ kg})\left(6at\hat{i}\right) = 12at\hat{i} \text{ kg}$$

$$F\big|_{t=2\text{ s}} = 48\hat{i} \text{ N}$$

$$\tau\big|_{t=2\text{ s}} = \vec{r}\big|_{t=2\text{ s}} \times \vec{F}\big|_{t=2\text{ s}} = \left(16\hat{i} + 8\hat{j} + 8\hat{k}\right) \times 48\hat{i} \text{ N m} = \left(384\hat{j} - 384\hat{k}\right) \text{ N m}$$

Example: A uniform lever of length 4 m is pivoted at its midpoint. The following forces are acting on the lever: A 5 N force (with a down ward vertical component) that makes an angle of 37° with the horizontal-left acting at the right end of the lever, A 4 N vertically down ward force acting 1 m to the left of the pivot, and a 2 N force whose direction is left acting at the left end of the lever. Calculate the net torque acting on the lever.

Solution: The weight of the lever does not contribute to the torque because it is uniform and it is pivoted at its midpoint.

$\vec{F}_1 = \left(-4\hat{i} - 3\hat{j}\right)$ N; $\vec{r}_1 = 2\hat{i}$ m; $\vec{F}_2 = -4\hat{j}$ N; $\vec{r}_2 = -\hat{i}$ m; $\vec{F}_3 = -2\hat{i}$ m; $\vec{r}_3 = -2\hat{i}$ m; $\vec{\tau} = ?$

$$\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1 = \left(2\hat{i}\text{ m}\right) \times \left(-4\hat{i} - 3\hat{j}\right) \text{ N} = -6\hat{k} \text{ N m}$$

$$\vec{\tau}_2 = \vec{r}_2 \times \vec{F}_2 = \left(-\hat{i}\text{ m}\right) \times \left(-4\hat{j}\text{ N}\right) = 4\hat{k} \text{ N m}$$

$$\vec{\tau}_3 = \vec{r}_3 \times \vec{F}_3 = \left(-2\hat{i}\text{ m}\right) \times \left(-2\hat{i}\text{ N}\right) = 0$$

$$\vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \tau_3 = (-6 + 4 + 0)\hat{k} \text{ N m} = -2\hat{k} \text{ Nm}$$

Since the direction is $-\hat{k}$, it is rotating in a clockwise direction.

11.3 RELATIONSHIP BETWEEN TORQUE AND ANGULAR ACCELERATION FOR A ROTATION ABOUT A FIXED AXIS

Consider a solid object rotating about a fixed axis (assumed to be along the z-axis) with an angular acceleration $\vec{\alpha} = \alpha_z \hat{k}$ under the influence of a force whose tangential component is F_θ. Let the tangential force acting on a small mass element of mass dm at a perpendicular distance $r_\perp$ from the axis of rotation be dF_θ. The torque acting on this mass element is $d\vec{\tau} = r_\perp dF_\theta \hat{k}$. From Newton's second law $dF_\theta = dma_\theta$ where a_θ is the tangential acceleration which is related with the angular acceleration by $a_\theta = r_\perp \alpha_z$. It follows that the torque acting on the mass element is given by $d\vec{\tau} = r_\perp{}^2 dm\alpha_z \hat{k}$. The total torque acting on the object is obtained adding (integrating) the torques on all the dm's: $\vec{\tau} = \hat{k}\int r_\perp{}^2 dm\alpha_z$. α_z can be taken out of the integral because all of the mass elements are rotating with the same angular acceleration. Therefore $\vec{\tau} = \alpha_z \hat{k}\int r_\perp{}^2 dm$. The integral $\int r_\perp{}^2 dm$ is equal to the moment inertia (I) of the object about the axis of rotation. The torque acting on the object is related with the angular acceleration of the object as

$$\vec{\tau} = I\alpha_z \hat{k} = I\vec{\alpha}$$

The direction of the torque is along the positive z-axis when α_z is positive and along the negative z-axis when α_z is negative. α_z is positive when it is either rotating in a counterclockwise direction and accelerating or rotating in a clockwise direction and decelerating; and it is negative when it is rotating in a clockwise direction and accelerating or rotating in a counterclockwise direction and decelerating. This equation is the equivalent of Newton's second law to rotation.

Example: The angular speed of a sphere of radius 2 cm and mass 0.5 kg rotating about an axis passing through its center increased from 2 rad/s to 14 rad/s uniformly in 3 seconds. Calculate the torque acting on it (The moment of inertia of a sphere of radius *R* and mass *M* about an axis passing through its center of mass is given by $I_{CM} = \frac{2}{5}MR^2$).

Solution:
$R = 0.02$ m; $M = 0.5$ kg; $\omega_{zi} = 2$ rad/s; $\omega_{zf} = 14$ rad/s; $t = 3$ s; $\alpha_z = ?$

$$I = I_{CM} = \frac{2}{5}MR^2 = \frac{2}{5}\times 0.5\times 0.02^2 \text{ kg m}^2 = 8\times 10^{-5} \text{ kg m}^2$$

$$\alpha_z = \frac{\omega_{zf} - \omega_{zi}}{t} = \frac{14-2}{3} \text{ rad/s}^2 = 4 \text{ rad/s}^2$$

$$\tau_z = I\alpha_z = 8\times 10^{-5}\times 4 \text{ N m} = 3.2\times 10^{-4} \text{ N m}$$

Example: A cylindrical wheel and axel of radius 5 cm and mass 2 kg is rotating by means of a an object hanging from the wheel via a string wound on the wheel. The mass of the hanging object is 10 kg. Calculate the tension in the string and the acceleration of the hanging object (Moment of inertia of a cylinder of mass *M* and radius *R* is given by $I_{CM} = \frac{MR^2}{2}$).

Solution:
$R = 0.05$ m; $M = 2$ kg; $m = 10$ kg; $T = ?$; $a = ?$

Rotating cylindrical wheel: It is rotating due to the tangential tension $\left(\vec{T}\right)$ exerted by the string.

$$\vec{T} = T\hat{e}_\theta \Rightarrow T_\theta = T$$

$$\vec{r}_\perp = R\hat{e}_{r_\perp} \Rightarrow r_\perp = R = 0.05 \text{ m}$$

$$\tau_z = r_\perp T_\theta = (0.05 \text{ m})T$$

$$I = \frac{MR^2}{2} = \frac{2\times 0.05^2}{2} \text{ kg m}^2 = 2.5\times 10^{-4} \text{ kg m}^2$$

$$\vec{a} = a_\theta \hat{e}_\theta = a\hat{e}_\theta \Rightarrow a_\theta = a$$

$$\tau_z = I\alpha_z = I\frac{a_\theta}{r_\perp} = I\frac{a}{R} = \left(2.5\times10^{-4}\ \text{kg m}^2\right)\frac{a}{0.05\ \text{m}} = \left(0.05\ \text{kg m}\right)a$$

$$\tau_z = \left(0.05\ \text{m}\right)T \text{ and } \tau_z = \left(0.05\ \text{kg m}\right)a \Rightarrow T = a\ \text{kg}$$

Hanging object: The forces acting on the object are its weight $(\vec{w})$ and the tension in the string $(\vec{T})$.

$$\vec{w} = -m|g|\hat{j} = -10\times9.8\hat{j}\ \text{N} \Rightarrow w_y = -98\ \text{N}$$

$$\vec{T} = T\hat{j} \Rightarrow T_y = T$$

$$\vec{a} = -a\hat{j} \Rightarrow a_y = -a$$

$$w_y + T_y = ma_y \Rightarrow -98\ \text{N} + T = -\left(10\ \text{kg}\right)a$$

$$T = a\ \text{kg and } -98\ \text{N} + T = -\left(10\ \text{kg}\right)a \Rightarrow a = 9.1\ \text{m/s}^2 \text{ and } T = 9.1\ \text{N}$$

Example: A sphere of mass 10 kg is rolling down an inclined plane that makes an angle of 53° with the horizontal-left. Use equations of linear and rotational motion to calculate its acceleration and the force of friction. (The moment of inertia of a sphere of mass M and radius R about an axis that passes through its center of mass is given as $I_{CM} = \frac{2}{5}MR^2$).

Solution:
Linear motion: The forces acting on the rolling sphere are its weight $(\vec{w})$, friction $(\vec{f})$ and normal force. Let's use a coordinate system where the positive x-axis is in the direction of the acceleration (then the positive y-axis is perpendicularly out of the plane). Then the angle formed between the weight and the x-axis is 90°–53° in a clockwise direction.

$$m = 10\ \text{kg};\ \theta_w = -37°;\ \theta_f = 180°;\ \theta_N = 90°;\ \theta_a = 0; f = ?;\ a = ?$$

$$\vec{w} = m|g|\cos\left(\theta_w\right)\hat{i} + m|g|\sin\left(\theta_w\right)\hat{j} \Rightarrow w_x = 78.3\ \text{N and } w_y = -59\ \text{N}$$

$$\vec{f} = f\cos\left(\theta_f\right)\hat{i} + f\sin\left(\theta_f\right)\hat{j} \Rightarrow f_x = -f \text{ and} f_y = 0$$

$$\vec{N} = N\cos\left(\theta_N\right)\hat{i} + N\sin\left(\theta_N\right)\hat{j} \Rightarrow N_x = 0 \text{ and } N_y = N$$

$$\vec{a} = a\cos\left(\theta_a\right)\hat{i} + a\sin\left(\theta_a\right)\hat{j} \Rightarrow a_x = a \text{ and } a_y = 0$$

$$w_x + f_x + N_x = ma_x \Rightarrow 78.3\ \text{N} - f = \left(10\ \text{kg}\right)a$$

Rotational motion: The axis of rotation passes through the center of the sphere. The lines of actions of the weight and normal forces also pass through the center of the sphere. Therefore the torques due to the weight and the normal force are zero because the perpendicular distance between the axis of rotation and these forces is zero. Only friction contributes to the torque. Friction acts at the point of contact and thus the perpendicular distance between the axis of rotation and the point at which friction acts is equal to the radius.

$$\vec{f} = f\hat{e}_\theta \Rightarrow f_\theta = f$$

$$\vec{r}_\perp = R\hat{e}_{r_\perp} \Rightarrow r_\perp = R$$

$$\vec{a} = a\hat{e}_\theta \Rightarrow a_\theta = a$$

$$\tau_z = r_\perp f_\theta = Rf$$

$$I = \frac{2}{5}MR^2 = \frac{2}{5}(10\text{ kg})R^2 = (4\text{ kg})R^2$$

$$\tau_z = I\alpha_z = I\frac{a_\theta}{R} = \left(4R^2\text{ kg}\right)\frac{a}{R} = (4\text{ kg})Ra$$

$$\tau_z = fR \text{ and } \tau_z = 4Ra\text{ kg} \Rightarrow f = (4\text{ kg})a$$

$$78.3\text{ N} - f = (10\text{ kg})a \text{ and } f = (4\text{ kg})a \Rightarrow a = 5.6\text{ m/s}^2 \text{ and } f = 22.4\text{ N}$$

Example: A 4 kg object on a frictionless table is connected to a hanging 15 kg object via a string that passes through a frictionless pulley on the edge of the table. The pulley is a disc of radius 2 cm and mass 0.2 kg. Calculate the tension in the part of the string between the pulley and the object, the tension in the part of the string between the pulley and the hanging object and the acceleration of the system.

Solution: The tension in the string is not uniform throughout because the pulley cannot rotate unless there is a difference between the tensions on both sides of the pulley. The magnitude of the acceleration for both objects $\left(\vec{a}\right)$ is the same for both objects because they are moving together.

$m_1 = 15 \text{ kg}; m_2 = 4 \text{ kg}; M = 0.2 \text{ kg}; R = 0.02 \text{ m}; a = ?; T_1 = ?; T_2 = ?$

Hanging object: The forces acting on the hanging object are its weight $\left(\vec{w}_1\right)$ and the tension in the string$\left(\vec{T}_1\right)$.

$$\vec{w}_1 = -m_1 |g| \hat{j} \Rightarrow w_y = -147 \text{ N and } w_x = 0$$

$$\vec{T}_1 = T_1 \hat{j} \Rightarrow T_{1y} = T_1 \text{ and } T_{1x} = 0$$

$$\vec{a}_1 = -a_1 \hat{j} = -a\hat{j} \Rightarrow a_{1x} = 0 \text{ and } a_{1y} = -a$$

$$w_y + T_{1y} = m_1 a_{1y} \Rightarrow -147 + T_1 = -\left(15 \text{ kg}\right) a$$

Object on the table: The forces acting are its weight, normal force and the tension $\left(\vec{T}_2\right)$ in the string. The weight and the normal force balance each other; it is tension in the string only that contributes to its acceleration.

$$\vec{T}_2 = T_2 \hat{i} \Rightarrow T_{2x} = T_2 \text{ and } T_{2y} = 0$$

$$\vec{a}_2 = a_2 \hat{i} = a\hat{i} \Rightarrow a_{2x} = a_2$$

$$T_{2x} = m_2 a_{2x} \Rightarrow T_2 = \left(4 \text{ kg}\right) a$$

Pulley: The forces responsible for the torque rotating the pulley are $\vec{T}_1$ and $\vec{T}_2$ Their torques are opposite in direction.

$$\vec{T}_1 = T_1 \hat{e}_\theta \Rightarrow T_{1\theta} = T_1$$

$$\vec{T}_2 = -T_2 \hat{e}_\theta \Rightarrow T_{2\theta} = -T_2$$

$$\vec{a} = a\hat{e}_\theta \Rightarrow a_\theta = a$$

$$\vec{r}_\perp = R\hat{e}_{r_\perp} \Rightarrow r_\perp = R = 0.02 \text{ m}$$

$$\tau_{1z} = r_\perp T_{1\theta} = (0.02 \text{ m})T_1$$

$$\tau_{2z} = r_\perp T_\theta = -(0.02 \text{ m})T_2$$

$$\tau_z = \tau_{1z} + \tau_{2z} = (0.02 \text{ m})(T_1 - T_2)$$

$$I = \frac{1}{2}MR^2 = \frac{1}{2}\times 0.2\times 0.02^2 \text{ kg m}^2 = 4\times 10^{-5} \text{ kg m}^2$$

$$\tau_z = I\alpha_z = I\frac{a_\theta}{R} = I\frac{a}{R} = \left(4\times 10^{-5} \text{ kg m}^2\right)\frac{a}{0.02 \text{ m}} = \left(2\times 10^{-3} \text{ kg m}\right)a$$

$$\tau_z = (0.02 \text{ m})(T_1 - T_2) \text{ and } \tau_z = \left(2\times 10^{-3} \text{ kg m}\right)a \Rightarrow T_1 - T_2 = (0.1 \text{ kg})a$$

$$T_2 = (4 \text{ kg})a \text{ and } T_1 - T_2 = (0.1 \text{ kg})a \Rightarrow T_1 = (4.1 \text{ kg})a$$

$$-147 + T_1 = -(15 \text{ kg})a \text{ and } T_1 = (4.1 \text{ kg})a \Rightarrow a = 7.7 \text{ m/s}^2$$

$$T_1 = (4.1 \text{ kg})a \text{ and } a = 7.7 \text{ m/s}^2 \Rightarrow T_1 = 31.6 \text{ N}$$

$$T_2 = (4 \text{ kg})a \text{ and } a = 7.7 \text{ m/s}^2 \Rightarrow T_2 = 30.8 \text{ N}$$

Practice Quiz 11.1

Choose the best answer

1. The SI unit of measurement for torque is
   A) Joule
   B) Joule * meter
   C) radian / second ^{2}
   D) Newton
   E) Newton * meter

2. A horizontal lever of length *2.2* m is pivoted at its mid-point. An upward force of *3* N is acting at the right end of the lever. Calculate the torque acting on the lever.
   A) *0.775* N m
   B) *5.896* N m
   C) *4.758* N m
   D) *5.125* N m
   E) *3.3* N m

3. A horizontal lever of length *1* m is pivoted at its mid-point. A *50* N downward force (force with downward vertical component) that makes an angle of *20* deg with the horizontal-left is acting at the right end of lever. Calculate the torque acting on the lever.
   A) *-6.377* N m
   B) *-4.571* N m
   C) *-7.516* N m
   D) *-8.551* N m
   E) *-11.889* N m

4. A uniform horizontal lever of length *2* m is pivoted at its mid-point. The following forces are acting on the lever: a *0.73* N vertically downward force acting at the right end of the lever, a *9* N vertically downward force acting at the left end of the lever, and a *8* N vertically downward force acting at a point on the lever *0.2* m away from the left end. Calculate the net torque acting on the lever.
   A) *14.67* N m
   B) *3.741* N m
   C) *1.726* N m
   D) *18.571* N m
   E) *16.89* N m

5. A uniform horizontal lever of length *4.8* m is pivoted at its mid-point. The following forces are acting on the lever: a *0.23* N downward force (with downward vertical component) that makes an angle of *40* deg acting at the right end of the lever, a *8* N vertically downward force acting at the left end of the lever, and a *2* N vertically downward force acting at a point on the lever *0.4* m away from the left end. Calculate the net torque acting on the lever.
   A) *40.068* N m
   B) *33.616* N m
   C) *17.912* N m
   D) *25.987* N m
   E) *22.845* N m

6. Calculate the torque acting on a uniform sphere of mass *3.7* kg and radius *0.24* m if it is rotating about an axis that passes through its center with an angular acceleration of *6* rad / s^2 ($I_{CM} = 2MR^2/5$)
   A) *0.152* Nm
   B) *0.349* Nm
   C) *0.614* Nm
   D) *0.895* Nm
   E) *0.511* Nm

7. A uniform disc of radius *0.57* m and mass *8* kg is pivoted at its center which is located at the origin of an xy-coordinate plane. The forces *8.3* N north, *14.3* N east, *8.3* N south and *15.6* N west are acting at the points (*0.57, 0*) m, (*0, 0.57*) m, (*-0.57, 0*) m and (*0, -0.57*) m respectively. Calculate the angular acceleration of the disc. ($I_{disc} = MR^2/2$)
   A) *-1.942* rad / s^2
   B) *-9.757* rad / s^2
   C) *-5.833* rad / s^2
   D) *-7.607* rad / s^2
   E) *-10.555* rad / s^2

8. A uniform rod of mass *4* kg and length *1.5* m is rotating about an axis that passes through the rod perpendicularly *0.3* m away from one of its end points. If, under the influence of some forces, its angular speed increased from *2.6* rad/s to *22.6* rad/s in *4* revolutions, calculate the torque acting on the sphere. ($I_{rod} = ML^2/12$)
   A) *12.378* Nm
   B) *18.849* Nm
   C) *8.108* Nm
   D) *15.642* Nm
   E) *24.397* Nm

9. An object of mass *3.2* kg is hanging from a cylindrical wheel and axle via a string. The wheel has a radius of *0.2* m and a mass of *2.6* kg. As the hanging object falls, calculate the tension in the string. ($I_{cylinder} = MR^2/2$)
   A) *9.06* N
   B) *4.118* N
   C) *14.979* N
   D) *5.949* N
   E) *10.183* N

10. A uniform spherical object of radius *0.2* m and mass *4.6* kg is rolling down an inclined plane of length *4* m and inclination *50*°. Calculate the force of friction. ($I_{CM} = 2MR^2/5$)
   A) *6.778* N
   B) *3.892* N
   C) *9.867* N
   D) *2.75* N
   E) *15.51* N

11. An object of mass *4.6* kg on a frictionless table is being pulled by a string attached to a hanging object of mass *14.8* kg via a frictionless cylindrical pulley of mass *5.2* kg and radius *0.02* m. Calculate the force exerted by the string on the sliding object object. ($I_{cylider} = MR^2/2$)
   A) *11.996* N
   B) *40.911* N
   C) *30.327* N
   D) *22.277* N
   E) *37.188* N

11.4 WORK DONE BY TORQUE FOR A ROTATION ABOUT A FIXED AXIS

Consider an object rotating about a fixed axis (assumed to be along the z-axis). Let dm be the mass of an arbitrary small mass element located at a perpendicular distance of $r_\perp$ from the axis of rotation. Let dF_θ be the tangential net force acting on this mass element. Then the work done by this force in displacing this mass element by an arc length of $ds = r_\perp d\theta_z$ is $dw(m,\theta_z) = dF_\theta r_\perp d\theta_z$. And with $dF_\theta = dma_\theta$ (Newton's second law), $dw(m,\theta_z) = dma_\theta r_\perp d\theta_z$. Again $a_\theta = r_\perp \alpha_z$ and thus $dw(m,\theta_z) = dmr_\perp{}^2 \alpha_z d\theta_z$. The amount of work done in rotating the whole object by an angle $d\theta_z$ can be obtained by integrating this expression over mass (α_z is the same for all of the mass elements and it can be factored out from the integral): $dw(\theta_z) = \alpha_z d\theta_z \int r_\perp{}^2 dm$. But the integral $\int r_\perp{}^2 dm$ is equal to the moment inertia (I) of the object around the axis of rotation. Therefore $dw(\theta_z) = I\alpha_z d\theta_z$. The product $I\alpha_z$ is equal to the torque (τ_z) acting on the object about the axis of rotation: $dw(\theta_z) = \tau_z d\theta_z$. The work done in displacing the object by a finite angle is obtained by integrating this angle over θ_z.

$$w = \int_{\theta_{zi}}^{\theta_{zf}} \tau_z d\theta_z$$

If the torque is a constant, then $w = \tau_z \int_{\theta_{zi}}^{\theta_{zf}} d\theta_z$ or

$$w = \tau_z \Delta\theta = \tau_z \left(\theta_{zf} - \theta_{zi}\right)$$

11.5 WORK-KINETIC ENERGY THEOREM FOR WORK DONE BY TORQUE

The net work done is the work done by the net torque. With $\tau_{net} = I\alpha_z = I\frac{d\omega_z}{dt}$ and $\frac{d\theta_z}{dt} = \omega_z$, $w_{net} = \int_{\theta_i}^{\theta_f} \tau_{net} d\theta = I\int_{\theta_i}^{\theta_f} \alpha_z d\theta_z = I\int_{\theta_i}^{\theta_f} \frac{dw_z}{dt} d\theta_z = I\int_{\omega_{zi}}^{\omega_{zf}} \omega_z d\omega_z$. Therefore, the net work done by a torque is related with the initial and final angular velocities as

$$w_{net} = \frac{1}{2}I\omega_f^{\,2} - \frac{1}{2}I\omega_i^2$$

This a mathematical statement of the work kinetic energy for rotational motion. The *work kinetic energy for rotational motion* states that the net work done by a torque is equal to the change in rotational kinetic energy: $w_{net} = \Delta KE_{rot}$.

Example: A disc of radius 0.2 m and mass 2 kg is rotating about an axle. Its angular speed decreased from 20 rad/s to 10 rad/s in 5 seconds because of the force of friction between the disc and the axle. The moment of inertia of a uniform disc of radius R and mass M about an axis through its center perpendicularly is given as $I_{CM} = \frac{MR^2}{2}$.

a) Calculate the work done by force of friction.
   Solution:
   $R = 0.2$ m; $M = 2$ kg; $\omega_{zi} = 20$ rad/s; $\omega_{zf} = 10$ rad/s; $t = 5$ s; $w_f = ?$

$$I = I_{CM} = \frac{MR^2}{2} = \frac{2\times 0.2^2}{2} \text{ kg m}^2 = 0.04 \text{ kg m}^2$$

$$w_f = \frac{1}{2}Iw_{zf}^{\,2} - \frac{1}{2}I\omega_{zi}^{\,2} = \frac{1}{2}I\left(\omega_{zf}^{\,2} - \omega_{zi}^{\,2}\right) = \frac{1}{2}\times 0.04\times\left(10^2 - 20^2\right) \text{ J} = -6 \text{ J}$$

b) If the radius of the axle is 0.01, calculate the force of friction.
   $R_{axle} = 0.01$ m; $f = ?$

$$\Delta\theta = \left(\frac{\omega_{zi} + \omega_{zf}}{2}\right)t = \left(\frac{20+10}{2}\right)5 \text{ rad} = 75 \text{ rad}$$

$$w_f = -6 \text{ J} = \tau_{fz}\Delta\theta = -fR_{axle}\Delta\theta = -f\left(0.01 \text{ m}\right)\left(75 \text{ rad}\right) = -0.75f \text{ m}$$

$$f = 8 \text{ N}$$

Example: A 2 kg object resting on the ground is connected to a hanging 10 kg object by a string via a pulley. The hanging object is 0.5 m above the ground. The pulley is a cylinder of radius 12 cm and mass 2 kg. If the 10 kg hanging object is released from rest, calculate its speed by the time it reaches the ground.

Solution: The only external force acting on the system (a system of two objects and a pulley) is gravity which is conservative. Therefore the sum of gravitational potential energy (including rotational kinetic energy of the pulley) is conserved. Let the subscripts '1' and '2' be used to identify the 10 kg and the 2 kg object respectively. Let the mass of the pulley be denoted by *M*. Both objects will have the same velocity at any time because they are connected together $\left(v_{1f} = v_{2f} = v_f \text{ and } v_{1i} = v_{2i} = v_i\right)$.

$m_1 = 10$ kg; $v_{1i} = 0$; $y_{1i} = 0$; $m_2 = 2$ kg; $v_{2i} = 0$; $y_{2i} = 0.5$ kg; $y_{2f} = 0.5$ m; $M = 2$ kg; $R = 0.12$ m; $v_{2f} = ?$

$$\frac{1}{2}m_1 v_i^2 + m_1|g|y_{1i} + \frac{1}{2}m_2 v_i^2 + m_2|g|y_{2i} + \frac{1}{2}I\omega_i^2 = \frac{1}{2}m_1 v_f^2 + m_1|g|y_{1f} + \frac{1}{2}m_2 v_f^2 + m_2|g|y_{2f} + \frac{1}{2}I\omega_f^2$$

$$\omega_i = \frac{v_i}{R} = 0,\ \omega_f = \frac{v_f}{R} \text{ and } y_{1i} = y_{2f} = y$$

$$m_1|g|y = \frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 v_f^2 + m_2|g|y + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v_f}{R}\right)^2$$

$$\left(m_1 - m_2\right)|g|y = \frac{1}{2}\left(m_1 + m_2\right)v_f^2 + \frac{1}{4}Mv_f^2$$

$$(10-2)(10)(0.5)\ \text{m}^2/\text{s}^2 = \frac{1}{2}(10+2)v_f^2 + \frac{1}{4}(2)v_f^2$$

$$v_f = \sqrt{\frac{80}{13}}\ \text{m/s}$$

11.6 ANGULAR MOMENTUM

Angular momentum $\left(\vec{L}\right)$ is a physical quantity used as a measure of rotational motion.

Angular momentum about a point: Angular momentum about a point is angular momentum where rotational motion about any axis is possible. The angular momentum of a particle of mass *m* moving with a velocity $\vec{v}$ when its position vector with respect to the point of rotation is $\vec{r}$ is defined to be the product of its mass and the cross product of its position vector and its velocity.

$$\vec{L} = m\vec{r} \times \vec{v}$$

This also may be written as $\vec{L} = \vec{r} \times m\vec{v}$ or

$$\vec{L} = \vec{r} \times \vec{p}$$

Where $\vec{p} = m\vec{v}$ is the linear momentum of the particle. Angular momentum of a particle is equal to the cross product between its position vector and its linear momentum. The magnitude of the angular momentum is equal to $rv\sin(\theta)$ where r, v and θ are the distance between the particle of the point of rotation and the particle, the speed of the particle and the angle between the position vector and the velocity vector respectively. The perpendicular distance $(r_\perp)$ between the point of rotation and the line of action of the velocity is equal to $r\sin(\theta)$. Therefore the magnitude of the angular momentum is given as

$$L = r_\perp v$$

The unit of measurement for angular momentum is kg m/s. The direction of angular momentum is perpendicular to the plane determined by the position vector and the velocity. In other words it is perpendicular to the plane of rotation. To distinguish between the two possible directions (perpendicularly out and perpendicularly into the plane of rotation), the screw rule or the right hand rule discussed on the chapter of vectors can be used. If the position vector and the velocity are known in terms of Cartesian components $\left(\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}\right.$ and $\left.\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right)$, then the angular momentum is given as

$$\vec{L} = \left(yv_z - zv_y\right)\hat{i} + \left(zv_x - xv_z\right)\hat{j} + \left(xv_y - yv_x\right)\hat{k}$$

Example: A particle of mass 0.1 kg is moving with a velocity $\left(3\hat{i}-6\hat{j}\right)$ m/s. Calculate its angular momentum when its location with respect to the point of rotation is $(4,6)$ m.

Solution:
$m = 0.1$ kg; $\vec{v}=\left(3\hat{i}-6\hat{j}\right)$ m/s; $\vec{r}=(4,6)$ m; $\vec{L}=?$

$$\vec{L}=m\vec{r}\times\vec{v}=3\left(4\hat{i}+6\hat{j}\right)\times\left(3\hat{i}-6\hat{j}\right)\text{ kg m/s}=-126\hat{k}\text{ kg m}^2/\text{s}$$

Example: The position vector of a particle of mass 0.1 kg varies with time according to the equation $\vec{r}=at^2\hat{i}-bt\hat{j}$ where $a=4$ m/s^2 and $b=2$ m/s. Calculate its angular moment time after 3 seconds.

Solution:
$m = 0.1$ kg; $\left.\vec{L}\right|_{t=3\text{ s}}=?$

$$\left.r\right|_{t=3\text{ s}}=\left(4\times3^2\hat{i}-2\times3\hat{j}\right)\text{ m}=\left(36\hat{i}-6\hat{j}\right)\text{ m}$$

$$\left.v\right|_{t=3\text{ s}}=\left.\frac{d\vec{r}}{dt}\right|_{t=3\text{ s}}=\left.\left(2at\hat{i}+b\hat{j}\right)\right|_{t=3\text{ s}}=\left(2\times4\times3\hat{i}-2\hat{j}\right)\text{ m/s}=\left(24\hat{i}-2\hat{j}\right)\text{ m/s}$$

$$\left.L\right|_{t=3\text{ s}}=m\left.\vec{r}\right|_{t=3\text{ s}}\times\left.\vec{v}\right|_{t=3\text{ s}}=0.1\left(36\hat{i}-6\hat{j}\right)\times\left(24\hat{i}-2\hat{j}\right)\text{ kg m}^2/\text{s}=7.2\hat{k}\text{ kg m}^2/\text{s}$$

Angular momentum about a fixed axis of rotation: Angular momentum about a fixed axis is angular momentum where only rotation about one axis is possible. Consider a particle of mass m moving with a velocity $\vec{v}$ when its position vector with respect to a point on the axis is $\vec{r}$. Let the position vector and the velocity be expressed as a sum of a component parallel to the axis of rotation and a component perpendicular to the axis of rotation: $\vec{r}=\vec{r}_\parallel+\vec{r}_\perp$ and $\vec{v}=\vec{v}_\parallel+\vec{v}_\perp$ ($\vec{r}_\perp$ and $\vec{v}_\perp$ lie on a plane perpendicular to the axis). Then $\frac{\vec{L}}{m}=\vec{r}\times\vec{v}=\left(\vec{r}_\parallel+\vec{r}_\perp\right)\times\left(\vec{v}_\parallel+\vec{v}_\perp\right)=\vec{r}_\parallel\times\vec{v}_\parallel+\vec{r}_\parallel\times\vec{v}_\perp+\vec{r}_\perp\times\vec{v}_\parallel+\vec{r}_\perp\times\vec{v}_\perp$. The first term is zero because the two vectors have the same line of action. The second and third terms are not possible because their directions are perpendicular to the fixed axis of rotation. The direction of the line of action of the fourth term is along the fixed axis of rotation because both vectors lie on a plane perpendicular to the axis of rotation; which means this term is the only possibility. Therefore for angular momentum about a fixed is given by $\vec{L}=m\vec{r}_\perp\times\vec{v}_\perp$. $\vec{r}_\perp$ and $v_\perp$ can be written in terms of a radially outward unit vector $\left(\hat{e}_{r_\perp}\right)$ and a tangential unit vector ($\hat{e}_\theta$: a unit vector tangent to a circle concentric to the axis of rotation in a counterclockwise direction) as $\vec{r}_\perp=r_\perp\hat{e}_{r_\perp}$ and $\vec{v}_\perp=v_{r_\perp}\hat{e}_{r_\perp}+v_\theta\hat{e}_\theta$. Then $\vec{L}=mr_\perp v_\theta\left(\hat{e}_{r_\perp}\times\hat{e}_\theta\right)$. Using a coordinate system where the z-axis lies along the axis of rotation, $\hat{e}_{r_\perp}\times\hat{e}_\theta=\hat{k}$ and $\vec{L}=mr_\perp v_\theta\hat{k}$. But $v_\theta=r_\perp\omega_z$ where ω_z is the angular velocity of the particle about the axis and $\vec{L}=mr_\perp{}^2\omega_z\hat{k}$ or

$$L_z=mr_\perp{}^2\omega_z$$

Where $r_\perp$ is the perpendicular distance between the axis of rotation and the particle. The expression $mr_\perp^{\,2}$ is the moment of inertia (I) of the particle about the x-axis of rotation. Thus, the angular momentum of a particle around a fixed axis may also be written as

$$L_z = I\omega_z$$

Now let's consider a solid object rotating about a fixed axis (assumed to be the z-axis) with an angular velocity ω_z. Let dm be the mass of an arbitrary small mass element at a perpendicular distance $r_\perp$ from the axis of rotation. If the mass element is moving with a velocity $v_\theta = r_\perp \omega_z$, then the angular momentum of this mass element is $dL_z = dmr_\perp^{\,2}\omega_z$ The total angular momentum is obtained by integrating over mass (the angular velocity is a constant for all the mass element): $L_z = \omega_z \int r_\perp^{\,2} dm.$ But the integral $\int r_\perp^{\,2} dm$ is equal to the moment of inertia (I) of the object about the axis of rotation. Therefore the angular momentum of an object rotating about a fixed axis with an angular velocity ω_z is given as

$$L_z = I\omega_z$$

Example: Calculate the angular momentum of a sphere of radius 4 cm and mass 4 kg (The moment of inertia of a sphere of radius R and mass M about an axis passing through its center is given as $I_{CM} = \frac{2}{5}MR^2$)

a) when it is rotating about an axis passing through its center with an angular velocity of 5 rad/s.
   Solution:
   $M = 4$ kg; $R = 0.04$ m; $\omega_z = 5$ rad/s; $L_z = ?$

$$I = I_{CM} = \frac{2}{5}MR^2 = \frac{2}{5}\times 4\times 0.04^2 \text{ kg m}^2 = \frac{128}{5} \text{ kg m}^2$$

$$L_z = I\omega_z = \frac{128}{5}\times 5 \text{ kg m}^2/\text{s} = 128 \text{ kg m}^2/\text{s}$$

b) When it is rotating about an axis tangent to the sphere with an angular speed of 2 rad/s.
   Solution: According to the parallel axis theorem, the moment of inertia about an axis that is a perpendicular distance d from a parallel axis that passes through the center of mass is given as $I = I_{CM} + Md^2$. In this case $d = R$.
   $\omega_z = 2$ rad/s; $L_z = ?$

$$I = I_{CM} + Md^2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$

$$L_z = I\omega_z = \frac{7}{5}MR^2\omega_z = \frac{7}{5}(4)(.04)^2(2) \text{ kg m}^2/\text{s} = \frac{896}{5} \text{ kg m}^2/\text{s}$$

11.7 CONSERVATION OF ANGULAR MOMENTUM

The rate change of angular momentum of an object is related with the torque acting on the object: $\frac{d\vec{L}}{dt} = m\frac{d}{dt}(\vec{r}\times\vec{v}) = m\frac{d\vec{r}}{dt}\times\vec{v} + m\vec{r}\times\frac{d\vec{v}}{dt}$. The first term is zero because $\frac{d\vec{r}}{dt} = \vec{v}$ and $\vec{v}\times\vec{v} = 0$. Thus $\frac{d\vec{L}}{dt} = m\vec{r}\times\frac{d\vec{v}}{dt} = \vec{r}\times m\frac{d\vec{v}}{dt}$. But $m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}$ which is the force acting on the particle. Therefore $\frac{d\vec{L}}{dt} = \vec{r}\times\vec{F}$ which is equal to the torque $(\vec{\tau})$ acting on the object. It follows that the torque acting on an object is equal to the rate of change of the angular momentum of the object.

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Now if the net torque on the object is zero, then $\int_{\vec{L}_i}^{\vec{L}_f} d\vec{L} = \vec{L}_f - \vec{L}_i = 0$. That is

$$\text{If } \vec{\tau}_{net} = 0, \text{ then } \vec{L}_f = \vec{L}_i$$

This is a mathematical statement of the principle of conservation of angular momentum. The *principle of conservation of angular momentum* states that if the net torque acting on an object is zero, then the angular momentum of the object is conserved. For a rotation about a fixed axis, since $\tau_z = \frac{dL_z}{dt} = \frac{d}{dt}(I_z \omega_z)$, it follows that if the net torque acting on the system is zero, then the product of the moment of inertia about the axis of rotation and the angular velocity remains constant.

$$I_{zf}\omega_{zf} = I_{zi}\omega_{zi}$$

Example: A skater of moment of inertia 5 kg m^2 is revolving with an angular speed of 3 rad/s. Calculate her new angular speed if she collapses her hands to reduce her moment of inertia to 4 kg m^2

Solution:
$I_{zi} = 5 \text{ kg m}^2$; $\omega_{zi} = 3$ rad/s; $I_{zf} = 4$ rad/s; $\omega_{zf} = ?$

$$I_{zf}\omega_{zf} = I_{zi}\omega_{zi}$$

$$\omega_{zf} = \frac{I_{zi}}{I_{zf}}\omega_{zi} = \frac{5}{4} \times 3 \text{ rad/s} = \frac{15}{4} \text{ rad/s}$$

Example: A rod of length 1m is lying on a frictionless table. One of its ends is pivoted so that it can be rotated freely. A bullet is fired to the rod perpendicularly as shown 0.25 m away from the free end with a speed of 500 m/s. The masses of the bullet and the rod are 0.4 kg and 5 kg,

Respectively. If the bullet is embedded in the rod, calculate the angular speed of the bullet-rod system after the collision.

Solution: Let's use a coordinate system where the y axis lies along the rod and the origin is located at the pivot.. Then the direction of the velocity of the bullet is in the $\hat{i}$ direction. Since only internal forces are involved, the angular momentum before and after the collision should be equal. Before the collision, only the bullet has angular momentum. The position vector of the point of impact is $\vec{r} = (1-0.25)\,\hat{j}$ m $= 0.75\hat{j}$ m. The velocity of the bullet is $\vec{v}_i = 500\hat{i}$ m/s. Therefore the angular momentum of the bullet before the impact is $\vec{L}_i = m\vec{r}\times\vec{v}_i = 0.4\left(0.75\hat{j}\times 500\hat{i}\right)$ kg m^2/s $= -150\hat{k}$ kg m^2/s. after the impact, the bullet rod system will rotate in a clockwise direction (and hence the direction is in the $-\hat{k}$ direction). Therefore the angular momentum after the impact is $\vec{L}_f = -I_z\omega_z\hat{k}$ where $I_z = I_{zb} + I_{zr}$ is the moment of inertia of the rod-bullet system which is the sum of the moment inertias of the bullet and the rod about the axis of rotation. Treating the bullet like a particle $I_{zb} = mr^2 = 0.4(0.75)^2$ kg m$^2 = 0.225$ kg m^2. Using the parallel axis theorem, it can be shown that the moment of inertia of a rod of length L and mass M is $\frac{ML^2}{3}$: $I_{zr} = \frac{5\times 1^2}{3}$ kg m$^2 = 1.667$ kg m^2. Thus $I_z = I_{zb} + I_{zr} = (1.667 + 0.225)$ kg m^2 $= 1.892$ kg m^2. Equating the angular momentums before and after the impact $150 = 1.892\omega_z$ or $\omega_z = 79.281$ rad/s.

Practice Quiz 11.2

Choose the best answer

1. The principle of conservation of angular momentum states that
   A) The angular momentum of an object will be conserved if the net force acting on the object is a constant.
   B) The angular momentum of an object will be conserved if the net force acting on the object is zero.
   C) The angular momentum of an object will be conserved if the net torque acting on the object is a constant.
   D) The angular momentum of an object will be conserved if all the forces acting on the object are conservative.
   E) The angular momentum of an object will be conserved if the net torque acting on the object is zero.

   Answer: E

2. A uniform rod of mass *2.4* kg and length *5.4* m is rotating about an axle passing through its center. Initially rotating with an angular speed of *12.7* rad/s was brought to rest in *2* revolutions because of the friction between the axle and the disc. If the radius of the axle is *0.041* m, calculate the force of friction between the axle and the disc ($I_{CM} = ML^2/12$)
   A) *696.49* N
   B) *385.456* N
   C) *912.854* N
   D) *1047.168* N
   E) *1550.766* N

Answer: C

3. Two objects of masses *7.2* kg and *18.6* kg are hanging on the opposite sides of a frictionless pulley of radius *0.2* m and mass *7.2* kg connected by a string. Initially, the *7.2* kg object is resting on the ground and the *18.6* kg object is *0.31* m above the ground. Then, the *18.6* kg object is released to fall. Calculate the speeds of the objects just before the *18.6* kg object hits the ground. ($I_{CM} = MR^2/2$)
   A) *0.456* m/s
   B) *0.88* m/s
   C) *1.535* m/s
   D) *2.661* m/s
   E) *1.149* m/s

   Answer: C

4. The position vector of a particle of mass *8.5* kg varies with time according to the equation $\boldsymbol{r} = 13.8t^2\ \boldsymbol{i} + 9.4t\ \boldsymbol{j}$ Calculate its angular momentum with respect to the origin after *6.3* seconds.
   A) *-37252.691* kg m^2/s $\boldsymbol{k}$
   B) *-74045.508* kg m^2/s $\boldsymbol{k}$
   C) *-50350.112* kg m^2/s $\boldsymbol{k}$
   D) *-65900.464* kg m^2/s $\boldsymbol{k}$
   E) *-43762.988* kg m^2/s $\boldsymbol{k}$

   Answer: E

5. A bullet of mass *0.057* kg is fired from the ground with a speed *312* m/s making an angle of *70°* with the horizontal-right. Calculate its angular momentum with respect to the point from which it was fired by the time it reaches its maximum height.
   A) *-5010.759* kg m^2/s $\boldsymbol{k}$
   B) *-9557.437* kg m^2/s $\boldsymbol{k}$
   C) *-40293.493* kg m^2/s $\boldsymbol{k}$
   D) *-26675.089* kg m^2/s $\boldsymbol{k}$
   E) *-47126.664* kg m^2/s $\boldsymbol{k}$

   Answer: D

6. Three particles of masses *2.3* kg, *9.5* kg and *10.7* kg are located at the points (*15.6, 5.9*) m, (*4, 1*) m, and (*17, -4*) m respectively. Calculate the angular momentum of this system of particles if they are revolving around the y-axis with an angular velocity of *24* rad / s.
   A) *54925.045* J s
   B) *91296.672* J s
   C) *44043.406* J s
   D) *123923.222* J s
   E) *157896.507* J s

   Answer: B

7. Calculate the angular momentum of a uniform rod of mass *4.9* kg and length *5.1* m if it is rotating about an axis that passes through its mid-point perpendicularly with an angular velocity of *5* rad / s ($I_{CM} = ML^2 / 12$)
   A) *44.227* kg m^2 / s
   B) *53.104* kg m^2 / s
   C) *66.345* kg m^2 / s
   D) *11.552* kg m^2 / s
   E) *73.225* kg m^2 / s

   Answer: B

8. A uniform disc of mass *7.2* kg and radius *0.12* is rotating about an axis that passes through the disc half way between its edge and its center perpendicularly with an angular velocity of *5.1* rad / s. Calculate its angular momentum. ($I_{CM} = MR^2 / 2$)
   A) *0.303* kg m^2 / s
   B) *0.536* kg m^2 / s
   C) *0.397* kg m^2 / s
   D) *0.225* kg m^2 / s
   E) *0.629* kg m^2 / s

   Answer: C

9. The angular momentum of a spherical object revolving about an axis passing through its center increased from *8* J s to *21* J s in *5* s. Calculate the torque acting on it.
   A) *2.6* N m
   B) *4.719* N m
   C) *0.735* N m
   D) *3.494* N m
   E) *1.77* N m

   Answer: A

10. Two particles of masses *4* kg and *13* kg located at $x = 1.3$ m and $x = -1.7$ m on the x-axis (connected by a rod of negligible mass) are revolving about the y-axis with an angular speed of *3.5* rad / s. If the *4* kg particle slides to the new location $x = 0.2$, Calculate the new angular speed with which the system is revolving.
    A) *5.969* rad / s
    B) *4.112* rad / s
    C) *5.495* rad / s
    D) *3.304* rad / s
    E) *7.087* rad / s

    Answer: B

11. A uniform disc of mass *2.4* kg and radius *0.92* m is rotating on a frictionless table about an axle that passes through its center with an angular speed of *5* rad / s. A second disc of mass *0.43* kg and radius *0.31* m is dropped on top of the rotating disc by sliding it through the axle via a hole on its center. Because of friction, both discs start rotating together. Calculate their angular speed. ($I_{CM} = MR^2/2$)
   A) *8.753* rad / s
   B) *4.9* rad / s
   C) *7.918* rad / s
   D) *3.729* rad / s
   E) *1.439* rad / s

Answer: B

12. A uniform rod of length *7.2* m and mass *9.5* kg is lying on a frictionless table. One of its ends is pivoted so that it can rotate freely. A bullet of mass *0.34* kg is fired to the free end of the rod perpendicularly (and parallel to the table) with a speed of *400* m / s. The bullet is embedded in the end of the rod and the rod-bullet system rotates about the pivot. Calculate the angular speed of the rod-bullet system after the collision. (Use the pivot as your origin and treat the bullet like a particle)($I_{CM} = ML^2/12$)
   A) *3.438* rad / s
   B) *4.768* rad / s
   C) *5.387* rad / s
   D) *9.852* rad / s
   E) *7.297* rad / s

Answer: C

12 STATIC EQUILIBRIUM

Your goal for this chapter is to learn how to solve problems involving objects in equilibrium.

There are two kinds of equilibrium: translational equilibrium and rotational equilibrium. An object is said to be in *translational equilibrium* if it is either at rest or moving in a straight line with a constant speed. An object will be in translational equilibrium if the net force acting on it is zero.

$$\sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots = 0$$

Or in component form

$$\sum F_x = F_{1x} + F_{2x} + F_{3x} + \ldots = 0$$
$$\sum F_y = F_{1y} + F_{2y} + F_{3y} + \ldots = 0$$

An object is said to be in *rotational equilibrium* if it is either at rest or rotating with a constant angular speed. An object is said to be in rotational equilibrium if the net torque acting on it is zero. For rotations about a fixed axis the torque is one dimensional and thus condition can be represented by a single component equation.

$$\sum \tau_z = \tau_{1z} + \tau_{2z} + \tau_{3z} + \ldots = 0$$

12.1 TORQUE DUE TO WEIGHT

The point at which an object can be balanced completely is called the center of gravity of the object (which is the same with the center of mass of the object). Consider an object balanced by applying an external force on its center of gravity. Since it is in equilibrium, the net torque about any point should be zero. Let's use a coordinate system where the object lies in the first quadrant and where the origin is the point of rotation. Let's assume the object is divided into small masses Δm_i. Then the torque due to the weight of each is Δm_i is $-\Delta m_i |g| x_i$. The torque due to the external balancing force is $M|g|x_{cm}$,, where M is the mass of the object. Since it is in equilibrium the torque due to its weight and the torque due to the external force should be equal.

$$\sum \Delta m_i |g| x_i = M |g| x_{cm}$$

Therefore it follows that the torque due to the weight of an object can be calculated by assuming the whole mass of the object is concentrated at its center of gravity.

Example: A uniform lever of length 5 m pivoted at its center is in equilibrium. An upward force of 4 N that makes an angle of 53° with the positive x-axis is acting at its right end. A vertically downward force of 20 N is acting at a point 2 m to the right of its pivot. An unknown vertically downward force is acting at a point a distance of 1.5 m to the left of the pivot. Assume the weight of the lever is negligible

a) Calculate the magnitude of the unknown force.
   Solution: The forces acting on the lever are the force exerted by the pivot and the listed forces. Thus there are two unknown forces (including the force due to the pivot). If the point of rotation is chosen at the point of rotation, the force due to the pivot will not contribute to the torques acting on the object. Thus, the application of the condition of rotational equilibrium with the pivot as point of rotation yields an equation with one unknown. Let the magnitude of the unknown force be denoted by F and the force due to the pivot be denoted by $\vec{F}_p$.

$$F_1 = 4\text{ N};\ \theta_1 = 53°;\ r_1 = 2.5\text{ m};\ F_2 = 20\text{ N};\ \theta_2 = 90°;\ r_2 = 2\text{ m};\ \theta_F = 90°;\ r_F = 1.5\text{ m};\ F = ?$$

$$\tau_1 + \tau_2 + \tau_F = 0$$

$$F_1 r_1 \sin(\theta_1) - F_2 r_2 \sin(\theta_2) + F r_F \sin(\theta_F) = 0$$

$$Fr_F \sin(\theta_F) = -F_1 r_1 \sin(\theta_1) + F_2 r_2 \sin(\theta_2) = \left(-4\times 2.5\sin(53°) + 20\times 2\times \sin(90°)\right) \text{ N m} = 32 \text{ N m}$$

$$F(1.5 \text{ m})\sin(90°) = 32 \text{ N m}$$

$$F = 21.3 \text{ N}$$

b) Calculate the x and y components of the force exerted by the pivot.
Solution: The x and y components of the force exerted by the pivot can be obtained by applying the condition of translational equilibrium in component form.

$$F_{1x} = 4\cos(53°) \text{ N} = 2.4 \text{ N};\ F_{1y} = 4\sin(53°) \text{ N} = 3.2 \text{ N};\ F_{2x} = 20\cos(-90°) = 0;\ F_{2y} = 20\sin(-90°) \text{ N} = -20 \text{ N};$$

$$F_x = 21.3\cos(-90°) = 0;\ F_y = 21.3\sin(-90°) \text{ N} = -21.3 \text{ N};\ F_{px} = ?;\ F_{py} = ?$$

$$F_{px} + F_{1x} + F_{2x} + F_x = 0$$

$$F_{px} + 2.4 + 0 + 0 = 0$$

$$F_{px} = -2.4 \text{ N}$$

$$F_{py} + F_{1y} + F_{2y} + F_y = 0$$

$$F_{py} + 3.2 \text{ N} + (-20 \text{ N}) + (-21.3 \text{ N}) = 0$$

$$F_{py} = 38.1 \text{ N}$$

Example: A uniform lever of length 2 m pivoted 0.5 m away from its left end is in equilibrium when the following forces are acting on it. $\vec{F}_1$: a 50 N vertically downward force acting on its left end, $\vec{F}_2$: a 5 N vertically downward force acting at its right end and $\vec{F}_3$: a vertically downward force acting 0.5 m away from its right end. Calculate the mass of the lever.

Solution: Since the lever is uniform, its center of gravity is located at its mid-point. The whole weight of the lever can be assumed to act at its midpoint. Let the mass of the lever be M and the torque due to its weight be τ_w.

$$\tau_{1z} = 0.5\times 50 \text{ N m} = 25 \text{ N m}$$

$$\tau_{2z} = -5\times 1.5 \text{ N m} = -7.5 \text{ N m}$$

$$\tau_{3z} = -1\times 8 \text{ N m} = -8 \text{ N m}$$

$$\tau_w = -(9.8 \text{ m/s}^2)(5 \text{ m})M = -49 \text{ N m/kg}$$

$$\tau_{1z} + \tau_{2z} + \tau_{3z} + \tau_w = 0 \Rightarrow M = 1.94 \text{ kg}$$

Example: A non-uniform of length 1 m and mass 3 kg pivoted at its midpoint is in equilibrium when acted upon by the following forces: 1. A 4 N vertically downward force acting 0.25 m away from its right end 2. A 2 N vertically downward force acting at its right end. Calculate the location of its center of gravity.

Solution: Let the location of the center of gravity be a distance of x to the left of the pivot. Since the lever is in equilibrium, the net torque acting on the lever should be zero. Let τ_w be the torque due to the weight. Taking the pivot to be the point of rotation

$$\tau_{1z} = -4(.25)\text{ N m} = -2\text{ N m}$$

$$\tau_{2z} = -2(0.5)\text{ N m} = -1\text{ N m}$$

$$\tau_w = (3\text{ kg})(9.8\text{ m/s}^2)x = 29.4x\text{ N}$$

$$\tau_{1z} + \tau_{2z} + \tau_w = 0 \Rightarrow x = 0.1\text{ m}$$

The center of gravity is located 0.1 m to the left of the pivot.

Practice Quiz 12.1

Choose the best answer

1. A uniform horizontal lever of length *4.33* m is pivoted at its mid-point. A vertically downward force of *9.71* N is acting at the left end of the lever. Where in the lever should a *14.33* N vertically downward force be applied if the lever is to be in equilibrium?
   A) *1.467* m to the right from the pivot
   B) *0.725* m to the left from the pivot
   C) *2.525* m to the right from the pivot
   D) *2.525* m to the left from the pivot
   E) *1.699* m to the left from the pivot

2. A uniform horizontal lever of length *3.23* m is pivoted at its mid-point. A vertically downward force of *9.71* N is acting at the right end of the lever. Calculate the vertically downward force that needs to be applied at a distance of *0.47* m to the left of the pivot if the lever is to be in equilibrium?
   A) *29.181* N
   B) *24.962* N
   C) *46.259* N
   D) *33.365* N
   E) *6.8* N

3. A uniform horizontal lever (with negligible weight) of length *4.83* m is pivoted at its mid-point. A vertically downward force of *7.71* N is acting at the right end of the lever. An unknown vertically downward force is acting at a distance of *0.63* m to the left of the pivot. If the lever is in equilibrium, calculate the force exerted by the pivot (fulcrum) on the lever.
   A) *60.761* N
   B) *53.913* N
   C) *10.667* N
   D) *37.265* N
   E) *5.123* N

4. A uniform horizontal lever of length *3.6* m pivoted at its mid-point is in equilibrium. The following forces are acting on the lever: a *3* N upward force (force with upward vertical component) that makes an angle of *30* deg with the horizontal-left acting at the right end of the lever, a *10* N vertically downward force acting at the left end of the lever, and an unknown vertically downward force acting at a point on the lever *0.2* m away from the right end. Calculate the magnitude of the unknown force.
   A) *2.365* N
   B) *12.937* N
   C) *10.524* N
   D) *17.432* N
   E) *23.205* N

5. A uniform horizontal lever of length *5.2* m pivoted at its mid-point is in equilibrium. The following forces are acting on the lever: a *4* N downward force (force with downward vertical component) that makes an angle of *80* deg with the horizontal-left acting at the right end of the lever, an unknown vertically downward force acting at the left end of the lever, and a *5* N vertically downward force acting at a point on the lever *0.2* m away from the right end. Calculate the magnitude of the unknown force.
   A) *5.901* N
   B) *14.66* N
   C) *13.705* N
   D) *9.752* N
   E) *8.555* N

6. A uniform horizontal lever of length *4* m pivoted at its mid-point is in equilibrium. The following forces are acting on the lever: an unknown upward force (force with upward vertical component) that makes an angle of *80* deg with the horizontal-left acting at the right end of the lever, a *8* N vertically upward force acting at the left end of the lever, and a/an *3* N vertically downward force acting at a point on the lever *0.1* m away from the right end. Calculate the magnitude of the unknown force.
   A) *1.325* N
   B) *11.017* N
   C) *16.551* N
   D) *14.849* N
   E) *19.393* N

7. A uniform lever of length *5* m is pivoted *1.93* m away from its right end. A vertically upward force of *14.9* N is acting *1.23* m away from its left end. If the lever is in equilibrium, calculate the mass of the lever.
   A) *0.985* kg
   B) *4.908* kg
   C) *5.989* kg
   D) *6.917* kg
   E) *7.789* kg

8. A uniform lever of length *4* m is pivoted *1.12* m from its right end. A downward force (with a downward vertical component) of *16.4* N that makes an angle of *50°* with the horizontal-left is acting at the right end of the lever. A vertically upward force of *12.6* N is acting at the left end of the lever. A vertically downward force of *5* N is acting *0.85* m away from the left end. If the lever is in equilibrium, calculate the mass of the lever.
   A) *6.385* kg
   B) *2.929* kg
   C) *0.741* kg
   D) *4.662* kg
   E) *8.04* kg

9. A non-uniform lever of mass *11.1* kg and of length *3* m is pivoted at its mid-point. The lever will be in equilibrium when an object of mass *9.1* kg hangs from its right end. Find the location of its center of gravity.
   A) *0.046* m from its left end
   B) *0.27* m from its left end
   C) *0.235* m from its left end
   D) *0.104* m from its left end
   E) *0.432* m from its left end

10. A non-uniform lever of mass *3* kg and length *4* m is pivoted at its mid-point. The lever will be in equilibrium when a vertically downward force of *2.3* N is acting at its right end and a vertically upward force of *5.6* N is acting on its left end. Calculate the location of its center of gravity.
   A) *0.433* m to the left of the pivot
   B) *0.783* m to the left of the pivot
   C) *0.537* m to the left of the pivot
   D) *0.855* m to the left of the pivot
   E) *0.653* m to the left of the pivot

Example: A uniform lever of mass 2 kg and length 2 m is hinged (its left end) to a wall and supported horizontally by means of a string connecting its right end to the wall. The string makes an angle of 37° with the horizontal lever. There is a weight of mass 1 kg hanging at a distance of 1.5 m from the hinge.

Solution: The weight of the lever acts at the center of gravity of the lever which is 1m away from the hinge. Taking the hinge as the point of rotation the torques due to the 1 kg weight and the weight of the lever are negative because they have clockwise tendencies. The torque due to the tension $(\vec{T})$ is positive because it has counterclockwise tendency.

$$\sum \tau_z = 0 \Rightarrow T(2\text{ m})\sin 37° - (2\text{ kg})(9.8\text{ m/s}^2)(1\text{ m}) - (1\text{ kg})(9.8\text{ m/s}^2)(1.5\text{ m}) = 0$$

$$T = 24.5\text{ N}$$

b) Calculate the force exerted by the hinge on the lever.
Solution: Let the force exerted by the hinge be represented by $\vec{F}_w$.

$$\sum F_x = 0 \Rightarrow F_{wx} - T\cos 37° = 0 \Rightarrow F_{wx} = 19.6 \text{ N}$$

$$\sum F_y = 0 \Rightarrow F_{wy} + T\sin 37° - (2 \text{ kg})(9.8 \text{ m/s}^2) - (1 \text{ kg})(9.8 \text{ m/s}^2) = 0 \Rightarrow F_{wy} = 14.7 \text{ N}$$

$$F_w = \sqrt{F_{wx}^2 + F_{wy}^2} = \sqrt{19.6^2 + 14.7^2} \text{ N} = 24.5 \text{ N}$$

$$\theta_{F_w} = \tan^{-1}\left(\frac{F_{wy}}{F_{wx}}\right) = \tan^{-1}\left(\frac{14.7}{19.6}\right) = 36.9°$$

Example: A uniform ladder of mass 4 kg and length 6 m is leaning on a frictionless wall. The ladder makes an angle of 37° with the horizontal-left on the ground.

a) Calculate the force exerted by the wall on the ladder.
Solution: Let the forces exerted by the wall, by the ground and by the weight of the ladder be represented by $\vec{F}_w$, $\vec{F}_G$ and $\vec{w}$ respectively. Since the wall is frictionless, the force exerted by the van on the ladder is only a normal force and is perpendicular to the wall. Taking the point of rotation to be the point of contact with the ground.

$$\tau_{F_w} = -F_w(6 \text{ m})\sin(37°) = -(3.6 \text{ m})F_w$$

$$\tau_w = (4 \text{ kg})(9.8 \text{ m/s}^2)(3 \text{ m})\sin(53°) = 94 \text{ N m}$$

$$\tau_{F_w} + \tau_w = 0 \Rightarrow F_w = 26 \text{ N}$$

b) Calculate the force exerted by the ground on the ladder.
Solution:

$$F_{wx} + F_{Gx} + w_x = 0 \Rightarrow -26 \text{ N} + F_{Gx} + 0 = 0 \Rightarrow F_{Gx} = 26 \text{ N}$$

$$F_{wy} + F_{Gy} + w_y = 0 \Rightarrow 0 + F_{Gy} - (4 \text{ kg})(9.8 \text{ m/s}^2) = 0 \Rightarrow F_{Gy} = 39.2 \text{ N}$$

$$F_G = \sqrt{F_{Gx}^2 + F_{Gy}^2} = \sqrt{(-26)^2 + (39.2)^2} \text{ N} = 47 \text{ N}$$

$$\theta_G = \tan^{-1}\left(\frac{F_{Gy}}{F_{Gx}}\right) = \tan^{-1}\left(\frac{39.2}{-26}\right) + 180° = 123.6°$$

Example: A uniform rod of length 3 m and mass 12 kg is suspended in a horizontal position by two strings connecting its ends to a ceiling. The string connected to its left end makes an angle of 53° with the horizontal-left at the ceiling. The string connected to its right end makes an angle 37° with the horizontal-right at the ceiling. A 10 kg object is hanging from the rod 0.5 m away from the left end.

a) Calculate the tension in the string attached to the right end.
Solution: Let the tensions in the strings attached to the right end and to the left end be represented by $\vec{T}_1$ and $\vec{T}_2$ respectively. Let's take the point of rotation to be the left end so that the torque due to the tension in the string connected to the left end is zero.

$$\sum \tau_z = 0 \Rightarrow T_1 \sin 37°(3\text{ m}) - (10\text{ kg})(9.8\text{ m/s}^2)(0.5\text{ m}) - (12\text{ kg})(9.8\text{ kg/m}^2)(1.5\text{ m}) = 0 \Rightarrow T_1 = 125\text{ N}$$

b) Calculate the tension in the string attached to the left end.
Solution: Let's take the point of rotation to be the right end. Then the torque due to the tension in the string attached to the right end is zero.

$$\sum \tau_z = 0 \Rightarrow -T_2 \sin 53°(3\text{ m}) + (10\text{ kg})(9.8\text{ m/s}^2)(2.5\text{ m}) + (12\text{ kg})(9.8\text{ kg/m}^2)(1.5\text{ m}) = 0 \Rightarrow T_2 = 175.6\text{ N}$$

Example: A uniform rod of length 2 m and mass 4 kg is pivoted at the ground. It makes an angle of 37° with the horizontal-right at the ground. Its right end is supporting a hanging object of mass 100 kg while being pulled to the left by means of a string attached to a wall.

a) Calculate the tension in the string.
Solution: Let the force due to the weight of the rod, the force exerted by the ground, the force due to the weight of the hanging object and the tension in the string be represented by $\vec{w}$, $\vec{F}_g$, $\vec{F}_h$ and $\vec{T}$ respectively. Let's take the point of rotation to be at the pivot on the ground so that the torque due to the force exerted by the ground is zero.

$$\sum \tau_z = 0 \Rightarrow T \sin 37(2\text{ m}) - (100\text{ kg})(9.8\text{ m/s}^2)\sin(53°)(2\text{ m}) - (4)(9.8)\sin(53°)(1\text{ m}) = 0 \Rightarrow T = 133\text{ N}$$

b) Calculate the force exerted by the ground on the lever.

$$w_x + F_{gx} + F_{hx} + T_x = 0 \Rightarrow F_{gx} - 1333\text{ N} = 0 \Rightarrow F_{gx} = 1333\text{ N}$$

$$w_y + F_{gy} + F_{hy} + T_y = 0 \Rightarrow F_{gy} - 100 \times 9.8\text{ N} - 4 \times 9.8\text{ N} = 0 \Rightarrow F_{gy} = 1019\text{ N}$$

$$\vec{F}_g = (1333\hat{i} + 1019\hat{j})\text{ N}$$

Practice Quiz 12.2

Choose the best answer

1. A uniform lever of mass *2.3* kg and length *4* m is being suspended horizontally from the ceiling by means of two strings attached to the ends of the lever. The string attached to the right end makes an angle of *30*° with the horizontal-left on the ceiling. The string attached to the left end makes an angle of *45*° with the horizontal-right on the ceiling. An object of mass *14.8* kg is hanging from the lever *0.5* m away from the right end. Calculate the tension of the string attached to the right end.
   A) *276.36* N
   B) *179.408* N
   C) *386.702* N
   D) *423.53* N
   E) *213.804* N

2. A uniform lever of mass *1.4* kg and length *4* m is being suspended horizontally from the ceiling by means of two strings attached to the ends of the lever. The string attached to the right end makes an angle of *50*° with the horizontal-left on the ceiling. The string attached to the left end makes an angle of *55*° with the horizontal-right on the ceiling. An object of mass *14.8* kg is hanging from the lever *0.5* m away from the right end. Calculate the tension of the string attached to the left end.
   A) *51.202* N
   B) *18.786* N
   C) *30.507* N
   D) *47.035* N
   E) *26.807* N

3. A uniform lever of mass *4.1* kg and length *3* m is hinged on a wall and is being suspended horizontally by means of a string attached to its free end and the wall. The string makes an angle of *60*° with the vertical-down on the wall. An object of mass *14.8* kg is hanging on the lever *0.6* m away from the hinge (wall). Calculate the tension in the string.
   A) *43.643* N
   B) *139.542* N
   C) *108.85* N
   D) *98.196* N
   E) *9.925* N

4. A uniform lever of mass *1.4* kg and length *3* m is hinged on a wall and is being suspended horizontally by means of a string attached to its free end and the wall. The string makes an angle of *60*° with the vertical-down on the wall. An object of mass *12.3* kg is hanging on the lever *0.6* m away from the hinge (wall). Calculate the direction of the force exerted by the wall on the lever.
   A) *82.16*°
   B) *110.385*°
   C) *50.731*°
   D) *62.558*°
   E) *38.865*°

5. A uniform ladder of mass *2.3* kg and length *4* m is leaning on a frictionless wall. The ladder makes an angle of *30°* with the horizontal-left on the ground. A man of mass of *100* kg is standing on the ladder *3* m away (along the ladder) from the ground. Calculate the force exerted by the wall on the ladder.
   A) *1292.578* N
   B) *1769.077* N
   C) *493.972* N
   D) *2056.547* N
   E) *732.238* N

6. A uniform ladder of mass *5.5* kg and length *4* m is leaning on a frictionless wall. The ladder makes an angle of *50°* with the horizontal-left on the ground. A man of mass of *80* kg is standing on the ladder *3* m away (along the ladder) from the ground. Calculate the direction of the force exerted by the ground on the ladder.
   A) *199.707°*
   B) *32.943°*
   C) *152.806°*
   D) *72.385°*
   E) *121.626°*

7. A uniform lever of mass *2.3* kg and length *5* m is pivoted at the ground. The lever makes an angle of *30°* with the horizontal-right on the ground while its top end is attached to a string that is attached to a nearby wall to the left of the lever horizontally. An object of mass *28.4* kg is hanging from the top of the lever. Calculate the tension in the string attached to the wall.
   A) *306.399* N
   B) *169.422* N
   C) *501.585* N
   D) *220.66* N
   E) *63.118* N

8. A uniform lever of mass *2.3* kg and length *5* m is pivoted at the ground. The lever makes an angle of *50°* with the horizontal-right on the ground while its top end is attached to a string that is attached to a nearby wall to the left of the lever horizontally. An object of mass *31.5* kg is hanging from the top of the lever. Calculate the direction of the force exerted by the ground on the lever.
   A) *13.582°*
   B) *42.812°*
   C) *94.027°*
   D) *50.973°*
   E) *67.682°*

13 SOLIDS AND FLUIDS

Your goal for this chapter is to learn about the effects of forces on solids and fluids.

13.1 SOLIDS

Solid is a state of matter with a fixed volume and fixed shape. The effect of force on a solid is either to change motion or shape. The change of motion aspect of it has been described in previous chapters. The change of shape aspect of it will be discussed in this chapter.

There are two physical quantities used to describe deformation of an object. These are stress and strain. *Stress* is a measure of a force's ability to deform an object. It is proportional to the magnitude of the force and inversely proportional to the area of the surface upon which the force is applied. It is defined to be the ratio between the force (F) and the area of the surface (A) over which the force is applied.

$$Stress = F/A$$

The unit of stress is N/m^2 which is defined to be the Pascal abbreviated as Pa. *Strain* is a physical quantity used as a measure of deformation. It is defined to be the ratio between the change and the original value. For example if the change is change in length, strain is defined to be ratio between the change in length and the original length. Since strain is ratio between the same physical quantities, it is unit-less.

Stress and strain are directly proportional as stress is increased from zero to a certain value. At a certain value the proportionality between stress and strain ceases to apply. This point where the proportionality between stress and strain breaks down is called the *elastic limit*. As the stress is increased further beyond the elastic limit, at a certain value the material breaks down. This point where the material breaks down is called the breaking point of the material. The constant of proportionality between stress and strain is called modulus.

$$modulus = stress/strain$$

The unit of measurement for modulus is Pascal.

There are three kinds of stresses. These are tensile stress, shear stress and bulk stress. *Tensile stress* is a stress where the force is applied parallel to the length of the material and perpendicular to the cross-sectional area of the material. The deformation is change in length and the strain is defined to be the ratio between the change in length (ΔL) and the original length (L). The modulus associated with this kind of stress is called *Young's modulus* (Y).

$$Y = (F/A)/(\Delta L/L)$$

Example: A steel wire of length *5* m is subjected to a force of *200* N. The cross-sectional radius of the wire is *0.003* m. Young's modulus for steel is *2e10* Pa. Calculate the change in its length.

Solution: $L = 5$ m; $F = 200$ N; $Y = 2e10$ Pa; $r = 0.003$ m; $\Delta L = ?$

$$A = \pi r^2 = 3.14 * 0.003^2 \text{ m}^2 = 0.000028 \text{ m}^2$$

$$Y = (F/A)/(\Delta L/L)$$

$$\Delta L = FL/YA = 200 * 5/(2e10 * 0.000028) \text{ m} = 0.0018 \text{ m}$$

Shear stress is a stress where the force is applied parallel to the surface. The effect of this stress is to produce deformation parallel to the surface. Its strain is defined to be the ratio between the deformation parallel to the surface (x) and the height (h) of the material in a direction perpendicular to this surface. The modulus associated with this kind of stress is called shear modulus (S).

$$S = (F/A)/(x/h)$$

Bulk stress is a stress where the stress is applied perpendicularly over the entire surface area of an object. The effect of this stress is to bring change in volume and its strain is defined to be the ratio between the change in volume (ΔV) and the original volume (V). The modulus associated with this kind of stress is called bulk modulus (B).

$$B = (-F/A)/(\Delta V/V)$$

The negative is introduced to make the bulk modulus positive, since the change in volume is negative; that is a decrease.

Example: A spherical solid aluminum ball of radius 0.002 m is subjected to a force of 2000 N applied throughout its surface area perpendicularly. The bulk modulus for aluminum is $7e10$ Pa. Calculate the change in its volume.

Solution: For a sphere of radius r, the volume and surface area are given by $4\pi r^3/3$ and $4\pi r^2$ respectively.

$F = 2000$ N; $B = 7e10$ Pa; $r = 0.002$ m; $\Delta V = ?$

$$A = 4\pi r^2 = 4 * 3.14 * 0.002^2 \text{ m}^2 = 0.00005 \text{ m}^2$$

$$V = 4\pi r^3/3 = 4 * 3.14 * 0.002^3/3 \text{ m}^3 = 3.3e\text{-}8 \text{ m}^3$$

$$B = (F/A)/(\Delta V/V)$$

$$\Delta V = FV/(BA) = 2000 * 3.3e\text{-}8/(7e10 * 5e\text{-}5) \text{ m}^3 = 1.9e\text{-}11 \text{ m}^3$$

13.2 FLUID STATICS

Fluid is a state of matter with a fixed volume but not fixed shape. It takes the shape of its container. The *density* (ρ) of fluid is defined to be the ratio between its mass (*m*) and its volume (*V*).

$$\rho = m/V$$

The unit of measurement for density is kg/m^3.

Fluid statics is the study of fluids at rest. There cannot be shear stress on a fluid at rest because if there were, the molecules of the fluid would be moving. At any volume element (part) of the fluid the forces act over the entire surface area of the volume element perpendicularly. The force (*F*) per unit area (*A*) is called pressure (*P*).

$$P = F/A$$

Unit of measurement for pressure is Pascal. The pressure due to air molecules is called *atmospheric pressure*. The value of atmospheric pressure at sea level is equal to *1.013e5* Pa. Atmospheric pressure decreases with the increase of altitude. A unit of pressure called atm is defined to be equal to atmospheric pressure at sea level.

Dependence of Fluid Pressure on Depth: Let's consider a part of a fluid in rest in cylindrical shape with the base of the cylinder parallel to the surface of the fluid. Let the base area of the cylinder be A and its height h. Since the fluid is at rest, this cylinder is in equilibrium and hence the net force acting on it must be zero. The forces acting on the cylinder are its weight and the force due to pressure difference between that at the bottom surface (P) and that at the top surface (P_0). The direction of the force due to pressure difference must be upwards since the direction of weight is downwards. The force due to pressure difference is equal to $(P - P_o)A$. The weight of the cylinder is $m|g|$. Its mass is equal to the product of the density (ρ) of the fluid and its volume (V) and its volume is equal to the product of its base area (A) and its height (h). Therefore the weight of the cylinder is equal to $Ah\rho|g|$. Equating its weight to the force due to pressure difference, the following equation for the dependence of pressure on depth can be obtained.

$$P = P_o + \rho|g|h$$

P_o is pressure at the upper level and P is pressure at the lower level. h is the separation between both levels. Pressure increases with depth linearly. This shows that a pressure applied on the surface is transmitted throughout the enclosed fluid undiminished. This is a statement of *Pascal's principle* which states that an external pressure applied to an enclosed fluid is transmitted undiminished to all parts of the enclosed fluid. This property of a fluid can be used to multiply force. Consider a U-shaped tube with different cross-sectional areas on both sides. If an external pressure is applied to the bigger cross-sectional area tube, the same pressure will be transmitted to the small cross-sectional area tube. Since pressure is equal to the ratio of force to cross-sectional area, this means a greater force has to be exerted by the smaller cross-sectional area tube if the pressure is to remain the same.

Example: Calculate the pressure *5* m below the surface of an ocean. Assume the density of the ocean to be *1000* $\mathrm{kg/m^3}$

Solution: The upper surface of the ocean is subjected to air molecules. Therefore the pressure at the surface of an ocean is equal to atmospheric pressure at sea level.

$h = 5$ m; $P_o = 1.013e5$ Pa; $\rho = 1000\ \mathrm{kg/m^3}$; $P = ?$

$$P = P_o + \rho|g|h = (1.013e5 + 1000 * 9.8 * 5)\ \mathrm{Pa} = 150300\ \mathrm{kg/m^3}$$

Measuring Pressure: Pressure is measured by a device called *manometer*. A manometer is essentially a U shaped tube filled with a fluid (most of the time mercury). From the knowledge of the pressure on one side of the U tube and the difference of the fluid levels on both sides, an unknown pressure on the other side of the tube can be calculated. There are two types of manometers. These are the closed manometer and the open manometer.

An *open manometer* is a manometer with one of the sides of the U tube open (exposed to air molecules). Thus the pressure at the surface of the fluid on this side of the tube is equal to atmospheric pressure. The gas of unknown pressure is connected to the other side of the tube and the difference between the fluid levels on both sides measured. Let atmospheric pressure be represented by P_{at} and the unknown gas pressure be denoted by P_g. If the level of the fluid on the gas side is lower than the other side, then $P_g = P$ (pressure at lower level) and $P_{at} = P_o$ (pressure at higher level). Therefore,

$$P_g = P_{at} + \rho|g|h$$

Where ρ is the density of the fluid and h is the separation between the fluid levels on both sides. Similarly if the fluid level at the air side is lower than the fluid level at the gas side, $P_g = P_o$ and $P_{at} = P$; and thus

$$P_g = P_{at} - \rho|g|h$$

Example: A gas of unknown pressure is connected to an open manometer filled with mercury. The density of mercury is $13,600$ kg/m^3. Atmospheric pressure is 100 kPa.

a) If the air side fluid level is 0.02 m higher than the gas side fluid level, calculate the gas pressure.
   Solution: $h = 0.02$ m; $P_{at} = 100$ kPa $= 100,000$ Pa; $\rho = 13,600$ kg/m^3; $p_g = ?$

$$P_g = P_{at} + \rho|g|h = (100,000 + 13,600 * 9.8 * 0.02) \text{ Pa} = 102665.6 \text{ Pa}$$

b) If the air side fluid level is 0.03 m below the gas side fluid level, calculate the gas pressure.
   Solution: $h = 0.03$ m; $P_g = ?$

$$P_g = P_{at} - \rho|g|h = (100,000 - 13,600 * 9.8 * 0.03) \text{ Pa} = 96001.6 \text{ Pa}$$

A *closed manometer* is a manometer with one side closed. The pressure on the surface of the fluid on the closed side is approximately zero because it is made to be approximately vacuum. Thus the upper level pressure is zero and the lower level pressure is the gas pressure.

$$P_g = \rho|g|h$$

Example: A gas of unknown pressure is connected to a closed manometer. The gas side fluid level is 0.08 m below the closed side fluid level. The fluid is mercury.

Solution: $h = 0.08$ m; $\rho = 13,\ 600$ kg/m^3; $p_g = ?$

$$P_g = \rho|g|h = (13,\ 600 * 9.8 * 0.08)\ \text{Pa} = 10662.4\ \text{Pa}$$

Measuring Atmospheric Pressure: Atmospheric pressure is measured by a device called *barometer*. A barometer is essentially a closed manometer. A closed tube is made to be approximately vacuum and inserted in a dish of mercury. The pressure on the closed tube is approximately zero. The mercury in the dish is exposed to atmospheric pressure. Because of the pressure difference, the mercury rises to a height h. The upper level pressure is zero and the lower level pressure is atmospheric pressure (P_{at}). Therefore,

$$P_{at} = \rho|g|h$$

Example: To what height would a mercury barometer rise at sea level?

Solution: $P_{at} = 1.013e5$ Pa; $\rho = 1.36e4$ kg/m^3; $h = ?$

$$P_{at} = \rho|g|h$$

$$h = P_{at}/(|g|\rho) = 1.013e5/9.8/1.36e4 \text{ m} = 0.76 \text{ m}$$

Practice Quiz 13.1

Choose the best answer

1. The ratio between the stress acting on an object and the resulting strain is called
   A) modulus
   B) stress constant
   C) Hook's constant
   D) elastic limit
   E) force constant

2. The kind of stress where the force is applied perpendicular to the cross-sectional area and parallel to the length of the material is called
   A) parallel stress
   B) tensile stress
   C) bulk stress
   D) shear stress
   E) normal stress

3. An aluminum wire has a length of *4* m and a cross-sectional radius of *0.005* m. Calculate the change in its length when an object of weight *60* N hangs from the wire. (modulus = *7e10* Pa)
   A) *6.346e-5* m
   B) *4.365e-5* m
   C) *3.54e-5* m
   D) *6.907e-5* m
   E) *8.067e-5* m

4. A spherical solid aluminum ball of radius *0.03* m is compressed by a force that is applied perpendicularly throughout its entire surface area. If its volume changed by *8e-9* m^3, calculate the magnitude of the force. (Modulus = *2.5e10* Pa).
   A) *37466.698* N
   B) *26096.472* N
   C) *35071.803* N
   D) *20000* N
   E) *11926.554* N

5. A *700* N force is acting on a circular surface of radius *0.07* m perpendicularly. Calculate the pressure exerted by the force on the surface.
   A) *67834.827* Pa
   B) *45472.841* Pa
   C) *30394.84* Pa
   D) *78692.814* Pa
   E) *14552.737* Pa

6. Calculate the pressure at a point in an ocean *90* m below the surface of the ocean. (Assume the density of the ocean to be *1000* kg/m^3)
   A) *302794.204* Pa
   B) *983300* Pa
   C) *1679245.316* Pa
   D) *1306849.449* Pa
   E) *832512.553* Pa

7. In an open manometer filled with mercury (density = *13600* kg/m^3), the level of mercury column in the air side is *0.06* m higher than that in the gas side. Determine the pressure of the gas. Atmospheric pressure is *9.7e4* Pa.
   A) *6397.44* Pa
   B) *104996.8* Pa
   C) *7996.8* Pa
   D) *89003.2* Pa
   E) *7197.12* Pa

8. In an open manometer filled with mercury (density = *13600* kg/m^3), the level of mercury column in the air side is *0.02* m lower than that in the gas side. Determine the pressure of the gas. Atmospheric pressure is *9.9e4* Pa.
   A) *101665.6* Pa
   B) *96334.4* Pa
   C) *2932.16* Pa
   D) *2665.6* Pa
   E) *2132.48* Pa

9. In a closed manometer filled with mercury (density = *13600* kg / m ^{3}), the level of mercury column in the vacuum side is *0.04* m higher than that in the gas side. Determine the pressure of the gas. Atmospheric pressure is *9.9e4* Pa.
   A) *93668.8* Pa
   B) *5864.32* Pa
   C) *5331.2* Pa
   D) *4264.96* Pa
   E) *104331.2* Pa

10. To what height would a mercury (density = *13600* kg / m ^{3}) barometer rise at a place where atmospheric pressure is *5e4* Pa?
   A) *0.424* m
   B) *0.375* m
   C) *0.614* m
   D) *0.31* m
   E) *0.708* m

Archimedes' Principle: Archimedes' principle states that an object immersed in a fluid is exerted upon by an upward buoyant force equal to the weight of the displaced fluid. The buoyant force is the force due to the difference between the upward pressure (on bottom surface) and the downward pressure (on top surface) acting on the object.

Let's consider a cylindrical object of density ρ_o height h and base area A immersed in a fluid of density ρ_f. The buoyant force (B) is equal to $(P - P_o)A$ where P is the pressure on the bottom surface and P_o is pressure at the top surface. But $P - P_o = \rho_f\ |g|h$. Therefore the buoyant force is given by $B = Ah\rho_f\ |g|h$. The product Ah is equal to the volume V of the cylinder.

$$B = V\rho_f\ |g|$$

Since the product $V\rho_f$ is equal to the mass of the fluid with the same volume as the object, it follows that buoyant force is equal to the weight of the displaced fluid.

An object immersed in a fluid weighs less than it does in air because of the upward buoyant force exerted by the fluid. The weight inside a fluid (W_f) is equal to the difference between the weight in air (W_a) and buoyant force.

$$W_f = W_a - B$$

Its weight in air i equal to the product of its mass (m_o) and gravitational acceleration; and its mass is equal to the product of its density and its volume.

$$W_a = \rho_o V|g|$$

Now the weight in fluid can be expressed in terms of the densities by using expressions for the weight in air and buoyant force in terms of density.

$$W_f = (\rho_o - \rho_f) V|g|$$

Example: An object of density *4000* kg / m ^{3} and volume of *0.002* m ^{3} is immersed in water (density *1000* kg / m ^{3}).

a) Calculate the upward force exerted by the fluid on the object.
Solution: $V = 0.002$ m^3; $\rho_f = 1000$ kg/m^3; B = ?
$B = \rho_f V|g| = 1000 * 0.002 * 9.8$ N $= 19.6$ N

b) Calculate its weight in air.
Solution: $\rho_o = 4000$ kg/m^3; W_a = ?
$W_a = \rho_o V|g| = 4000 * 0.002 * 9.8$ N $= 79.4$ N

c) Calculate its weight inside the fluid.
Solution: W_f = ?
$W_f = W_a - B = (79.4 - 19.6)$ N $= 59.8$ N

For a floating object, the weight of the object in air and the buoyant force exerted by the fluid must balance each other, because the object is in equilibrium. Suppose a floating object of density ρ_o and volume V floats in a fluid of density ρ_f with V_i part of its volume immersed. Its weight in air is equal to $\rho_o V|g|$ and according to Archimedes' principle (buoyant force equals weight of displaced fluid) the buoyant force is equal to $\rho_f V_i |g|$. Equating the weight in air and the buoyant force, the following equation for floating objects is obtained.

$$\rho_o / \rho_f = V_i / V$$

The following observations can be made from this equation: 1) An object whose density is smaller than the density of the fluid floats partially immersed 2) An object whose density is equal to the density of the fluid floats with all of its volume immersed. And of course an object whose density is greater than that of the fluid sinks.

Example: An object floats in water (density $1000\ \mathrm{kg/m^3}$) with 25% of its volume immersed in the fluid. Calculate the density of the object.

Solution: $\rho_\mathrm{f} = 1000\ \mathrm{kg/m^3}$; $V_\mathrm{i} = 0.25V$; $\rho_\mathrm{o} = ?$

$$\rho_\mathrm{o}/\rho_\mathrm{f} = Vi/V$$

$$\rho_\mathrm{o}/(1000\ \mathrm{kg/m^3}) = 0.25V/V = 0.25$$

$$\rho_\mathrm{o} = 250\ \mathrm{kg/m^3}$$

13.3 FLUID DYNAMICS

Fluid dynamics is the study of fluids in motion.

Continuity Equation: The Continuity equation is a mathematical statement of the fact that the amount of fluid that enters a tube is equal to the amount of fluid that leaves the tube in the same interval of time. Suppose fluid enters a tube of cross-sectional area A_1 with speed v_1 and leaves a tube of cross-sectional area A_2 with speed v_2. In a time interval Δt, fluid of length $v_1 \Delta t$ enters tube 1 and fluid of length $v_2 \Delta t$ leaves tube 2. In other words, in a time interval Δt, fluid of volume $A_1 v_1 \Delta t$ enters tube 1 and fluid of volume $A_2 v_2 \Delta t$ leaves tube 2. Equating these two volumes, the equation so called continuity equation is obtained.

$$A_1 v_1 = A_2 v_2$$

Example: Two tubes of cross-sectional radii of *0.02* m and *0.04* m are connected together. Water enters the first tube with a speed of *20* m/s. Calculate the speed with which it will leave the second tube.

Solution: $v_1 = 20$ m/s; $r_1 = 0.02$ m ($A_1 = \pi r_2{}^2$); $r_2 = 0.04$ m ($A_2 = \pi r_2{}^2$); $v_2 = ?$

$$A_1 v1 = A_2 v_2$$

$$\pi r_1{}^2 v_1 = \pi r_2{}^2 v_2$$

$$(0.02 \text{ m})^2 (20 \text{ m/s}) = (0.04 \text{ m})^2 v_2$$

$$v_2 = 5 \text{ m/s}$$

Bernoulli's Equation: Let's consider two tubes, tube 1 and 2, connected together with the elevation of the first being y_1 and the elevation of the second being y_2. There are two kinds of forces acting on a fluid flowing through these tubes. These are gravity and the force due to pressure difference in the tubes. Gravity is a conservative force but the force due to pressure difference is non-conservative. The work done by a non-conservative force is equal to change in mechanical energy. If a part of the fluid of mass Δm is taken, the change in mechanical energy is $(\Delta m v_2{}^2/2 + \Delta m|g|y_2) - (\Delta m v_1{}^2/2 + \Delta m|g|y_1)$ and the work done by the force due to pressure difference is $(P_1 - P_2)\Delta V$ where ΔV is the volume of the fluid. Equating these two expressions, dividing the equation by ΔV and noting that $\Delta m/\Delta V$ is equal to the density of the fluid ρ, the following equation that is called Bernoulli's equation is obtained.

$$P_1 + \rho|g|y_1 + \rho v_1{}^2/2 = P_2 + \rho|g|y_2 + \rho v_2{}^2/2$$

Example: A tube of cross-sectional area 0.0004 m ^{2} is connected to a tube of cross-sectional area 0.0008 m ^{2}. The second tube is elevated 0.01 m higher than the first tube. Water enters the first tube with a speed of 10 m/s. The pressure on the first tube is $10, 000$ Pa.

a) Calculate the speed of the fluid in the second tube.
Solution: $v_1 = 10$ m/s; $A_1 = 0.0004$ m ^{2}; $A_2 = 0.0008$ m ^{2}; $v_2 = ?$

$$A_1 v_1 = A_2 v_2$$

$$(0.0004 \text{ m}^2)(10 \text{ m/s}) = (0.0008 \text{ m}^2) v_2$$

$$v_2 = 5 \text{ m/s}$$

b) Calculate the pressure on the second tube.
Solution: Let the origin of the coordinate system be fixed at the lower tube (tube 1).

$\rho = 1000$ kg/m ^{3}; $P_1 = 10\ 000$ Pa; $y_1 = 0$; $y_2 = 0.01$ m; $P_2 = ?$

$$P_1 + \rho|g|y_1 + \rho v_1{}^2/2 = P_2 + \rho|g|y_2 + \rho v_2{}^2/2$$

$$(10\ 000 \text{ Pa}) + (1000 \text{ kg/m}^3)(10 \text{ m/s})^2/2 = P_2 + (1000 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(0.01 \text{ m}) + (1000 \text{ kg/m}^3)(5 \text{ m/s})^2/2$$

$$P_2 = 47402 \text{ Pa}$$

Practice Quiz 13.2

Choose the best answer

1. Archimedes principle states that
   A) An object immersed in a fluid is acted upon by a downward force equal to the weight of the object.
   B) An object sinks in a fluid if its density is greater than the density of the fluid.
   C) the pressure exerted by a fluid increases with depth
   D) An object immersed in a fluid is acted upon by an upward force equal to the weight of the object.
   E) An object immersed in a fluid is acted upon by an upward force equal to the weight of the displaced fluid.

2. An object will float in a fluid partially immersed if
   A) its density is greater or equal to the density of the fluid
   B) its density is less or equal to the density of the fluid
   C) its density is equal to the density of the fluid
   D) Its density is greater than the density of the fluid.
   E) Its density is less than the density of the fluid.

3. An object of volume *2e-6* m^3 and density *8500* kg / m^3 is immersed inside a fluid of density *4000* kg / m^3. Calculate the weight of the object in the fluid (i.e., the reading of a spring balance attached to it when inside the fluid).
   A) *0.078* N
   B) *0.088* N
   C) *0.011* N
   D) *0.056* N
   E) *0.151* N

4. An object of volume *2e-4* m^3 is immersed in a fluid. If the fluid is exerting an upward force of *0.3* N on the object, calculate the density of the fluid.
   A) *37.98* kg/m^3
   B) *265.933* kg/m^3
   C) *64.88* kg/m^3
   D) *249.872* kg/m^3
   E) *153.061* kg/m^3

5. Calculate the density of an object that floats in a fluid of density *3500* kg/m^3 with *70* % of its volume immersed in the fluid.
   A) *3614.413* kg/m^3
   B) *4089.258* kg/m^3
   C) *2830.595* kg/m^3
   D) *2450* kg/m^3
   E) *1875.837* kg/m^3

6. An object of volume *7.5e-5* m^3 is floating in a fluid of density *9000* kg/m^3 with *2.5e-5* m^3 of its volume exposed above the surface of the fluid. Calculate the density of the object.
   A) *6000* kg/m^3
   B) *1454.016* kg/m^3
   C) *8117.699* kg/m^3
   D) *11269.613* kg/m^3
   E) *4076.869* kg/m^3

7. Tube *1* and tube *2* are connected together. A fluid flowing through these tubes has a speed of *8* m/s in tube *1* and a speed of *17* m/s in tube *2*. Calculate the ratio of the cross-sectional radius of tube *2* to the cross-sectional radius of tube *1*.
   A) *0.4*
   B) *0.965*
   C) *0.686*
   D) *1.266*
   E) *0.537*

8. A tube of cross-sectional-radius *0.07* m is connected with a tube of cross-sectional radius *0.06* m. If the speed of a fluid in the *0.07* m cross-sectional radius tube is *30* m / s, calculate the speed of the fluid in the *0.06* m cross-sectional radius tube.
   A) *26.8* m / s
   B) *20.701* m / s
   C) *34.435* m / s
   D) *47.554* m / s
   E) *40.833* m / s

9. Two tubes of different cross-sectional radii are connected horizontally. A fluid of density *3500* kg / m ^{3} enters the first tube with a speed of *2.25* m / s. The cross-sectional radius of the first tube is *0.08* m and that of the second tube is *0.14* m. If the pressure of the fluid in the first tube is *1e3* Pa, calculate the pressure of the fluid in the second tube.
   A) *10479.787* Pa
   B) *8914.769* Pa
   C) *5099.859* Pa
   D) *3518.035* Pa
   E) *12464.625* Pa

10. Two tubes of different cross-sectional radii are connected together with the first tube elevated *0.8* m above the second tube. The speed of a fluid of density *1750* kg / m ^{3} in the first and second tube respectively are *1.8* m / s and *0.8* m / s. If the pressure of the fluid in the first tube is *7e3* Pa, calculate the pressure of the fluid in the second tube.
    A) *40015.626* Pa
    B) *22995* Pa
    C) *15916.836* Pa
    D) *4394.337* Pa
    E) *12870.845* Pa

14 GRAVITATION

Your goal for this chapter is to learn about the properties of gravitational forces.

Newton's Law of gravitation states that any two objects in the universe attract each other with a gravitational force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers. The direction of the force is along the line joining their centers. The constant of proportionality between the gravitational force and the ratio of the product of the masses and the square of the distance separating them is called universal gravitational constant and denoted by G

$$G = 6.67 \times 10^{-11}\ \mathrm{N\ m^2/kg^2}$$

Consider the gravitational force exerted by an object of mass m_1 on an object of mass m_0. Let $\vec{r}_1$ and $\vec{r}_0$ be the position vectors of the locations of m_1 and m_0 with respect to a certain coordinate system respectively. Then, the position vector of the location of m_0 with respect to the location of m_1 ($\vec{r}_{01}$) is equal to the difference between $\vec{r}_0$ and $\vec{r}_1$; that is $\vec{r}_{01} = \vec{r}_0 - \vec{r}_1$. $\vec{r}_{01}$ is the vector whose tail is at the location of m_1 and whose head is at the location of m_0. Let $\hat{e}_{01}$ be a unit vector in the direction of $\vec{r}_{01}$. Then $\hat{e}_{01} = \frac{\vec{r}_{01}}{r_{01}} = \frac{\vec{r}_0 - \vec{r}_1}{|\vec{r}_0 - \vec{r}_1|}$. According to Newton's law of gravitation, the magnitude of the gravitational force (F_{01}) is proportional to the expression $\frac{m_0 m_1}{{r_{01}}^2}$.

$$F_{01} = G\frac{m_0 m_1}{{r_{01}}^2} = G\frac{m_0 m_1}{|\vec{r}_0 - \vec{r}_1|^2}$$

Since gravitational force is attractive, the direction of the force exerted on m_0 by m_1 is opposite to the direction of $\vec{r}_{01}$. In other words the direction of the force is $-\hat{e}_{01}$ Therefore the gravitational force can be written in vector form as $\vec{F}_{01} = -G\frac{m_0 m_1}{|\vec{r}_0 - \vec{r}_1|^2}\hat{e}_{01}$. But $\hat{e}_{01} = \frac{\vec{r}_0 - \vec{r}_1}{|\vec{r}_0 - \vec{r}_1|}$.

$$\vec{F}_{01} = -G\frac{m_0 m_1}{{r_{01}}^3}\vec{r}_{01} = -G\frac{m_0 m_1}{|\vec{r}_0 - \vec{r}_1|^3}(\vec{r}_0 - \vec{r}_1)$$

Example: An object of mass 10^3 kg is located 100 m to the right of an object of mass 10^6 kg

a) Calculate the gravitational force exerted by the 10^6 kg object on the object 10^3 kg.

Solution: Let's use a coordinate system where the x-axis lies along the line joining the two objects and the origin is at the location of the 10^6 kg. Remember the subscript zero is used to the object that is being acted upon.

$$m_0 = 10^3 \text{ kg};\ m_1 = 10^6 \text{ kg};\ \vec{r}_0 = 100\hat{i};\ \vec{r}_1 = 0;\ \vec{F}_{01} = ?$$

$$\vec{r}_0 - \vec{r}_1 = 100\hat{i} \text{ m} - 0 = 100\hat{i} \text{ m}$$

$$|\vec{r}_0 - \vec{r}_1| = 100 \text{ m}$$

$$\vec{F}_{01} = -G\frac{m_0 m_1}{|\vec{r}_0 - \vec{r}_1|^3}(\vec{r}_0 - \vec{r}_1) = -6.67\times10^{-11}\frac{10^3\times10^6}{100^3}(100\hat{i}) \text{ N} = -6.67\times10^{-6}\hat{i} \text{ N}$$

b) Calculate the gravitational force exerted by the 10^3 kg object on the 10^6 kg object.
Solution: Remember the subscript zero is used to the object that is being acted upon.

$$m_1 = 10^3 \text{ kg}; m_0 = 10^6 \text{ kg}; \vec{r}_1 = 100\hat{i}; \vec{r}_0 = 0; \vec{F}_{01} = ?$$

$$\vec{r}_0 - \vec{r}_1 = 0 - 100\hat{i} \text{ m} = -100\hat{i} \text{ m}$$

$$|\vec{r}_0 - \vec{r}_1| = 100 \text{ m}$$

$$\vec{F}_{01} = -G\frac{m_0 m_1}{|\vec{r}_0 - \vec{r}_1|^3}(\vec{r}_0 - \vec{r}_1) = -6.67\times10^{-11}\frac{10^3\times10^6}{100^3}\left(-100\hat{i}\right) \text{ N} = 6.67\times10^{-6}\hat{i} \text{ N}$$

As expected, the gravitational forces exerted on each other are equal in magnitude and opposite in direction because they are action reaction forces.

Superposition Principle: If an object is in the vicinity of more than one object, then the net gravitational force acting on the object is the vector sum of all the forces due to the objects in its vicinity. The gravitational force $\left(\vec{F}_0\right)$ exerted on an object of mass m_0 whose position vector is $\vec{r}_0$ due to objects of masses $m_1, m_2, m_3,...$ whose position vectors are $\vec{r}_1, \vec{r}_2, \vec{r}_3,...$ respectively is given by

$$\vec{F}_0 = -Gm_0\left(\frac{m_1}{|\vec{r}_0 - \vec{r}_1|^3}(\vec{r}_0 - \vec{r}_1) + \frac{m_2}{|\vec{r}_0 - \vec{r}_2|^3}(\vec{r}_0 - \vec{r}_2) + \frac{m_3}{|\vec{r}_0 - \vec{r}_3|^3}(\vec{r}_0 - \vec{r}_3) + ...\right) = \sum_i \frac{m_i}{|\vec{r}_0 - \vec{r}_i|^3}(\vec{r}_0 - \vec{r}_i)$$

Example: Particles of masses 100 kg, 200 kg and 300 kg are located at the points $(0,0)$ m, $(500,0)$ m and $(300,400)$ m respectively.

a) Calculate the net gravitational force exerted on the 100 kg particle by the 200 kg and the 300 kg particles.

Solution: Remember the subscript zero is used to the object that is being acted upon.

$$m_0 = 100 \text{ kg}; \vec{r}_0 = 0; m_1 = 200 \text{ kg}; \vec{r}_1 = (500,0) \text{ m} = 500\hat{i} \text{ m};$$

$$m_2 = 300 \text{ kg}; \vec{r}_2 = (300,400) \text{ m} = \left(300\hat{i} + 400\hat{j}\right) \text{ m}; \vec{F}_0 = ?$$

$$\vec{r}_0 - \vec{r}_1 = 0 - 500\hat{i} \text{ m} = -500\hat{i} \text{ m}$$

$$|\vec{r}_0 - \vec{r}_1| = 500 \text{ m}$$

$$\vec{r}_0 - \vec{r}_2 = 0 - \left(300\hat{i} + 400\hat{j}\right) \text{ m} = \left(-300\hat{i} - 400\hat{j}\right) \text{ m}$$

$$|\vec{r}_0 - \vec{r}_2| = \sqrt{(-300)^2 + (-400)^2} \text{ m} = 500 \text{ m}$$

$$\vec{F}_0 = -Gm_0\left(\frac{m_1}{|\vec{r}_0-\vec{r}_1|^3}(\vec{r}_0-\vec{r}_1)+\frac{m_2}{|\vec{r}_0-\vec{r}_2|^3}(\vec{r}_0-\vec{r}_2)\right)$$

$$= -6.67\times10^{-11}\times100\left(\frac{200}{500^3}(-500\hat{i})+\frac{300}{500^3}(-300\hat{i}-400\hat{j})\right)\text{ N}$$

$$= -6.67\times10^{-11}\times\frac{100}{500^3}\left(-1.9\times10^5\hat{i}-1.2\times10^5\hat{j}\right)\text{ N}$$

$$\left(1.58\times10^{-7}\hat{i}+10^{-7}\hat{j}\right)\text{ N}=\left(1.58\hat{i}+\hat{j}\right)10^{-7}\text{ N}$$

b) Calculate the net gravitational force exerted on the 300 kg particle by the 100 kg and the 200 kg particles.

Solution: Remember the subscript zero is used to the object that is being acted upon.
$m_0 = 300$ kg; $\vec{r}_0 = (300, 400)$ m $= \left(300\hat{i}+400\hat{j}\right)$ m; $m_1 = 100$ kg; $\vec{r}_1$=0;

$m_2 = 200$ kg; $\vec{r}_2 = (500, 0)$ m $= \left(500\hat{i}\right)$ m; $\vec{F}_0 = ?$

$$\vec{r}_0-\vec{r}_1 = \left(300\hat{i}+400\hat{j}\right)\text{ m}-0=\left(300\hat{i}+400\hat{j}\right)\text{ m}$$

$$|\vec{r}_0-\vec{r}_1| = \sqrt{(300)^2+(400)^2}\text{ m}=500\text{ m}$$

$$\vec{r}_0 - \vec{r}_2 = \left(300\hat{i} + 400\hat{j}\right)\text{ m} - \left(500\hat{i}\right)\text{ m} = \left(-200\hat{i} + 400\hat{j}\right)\text{ m}$$

$$\left|\vec{r}_0 - \vec{r}_2\right| = \sqrt{(-200)^2 + (400)^2}\text{ m} = 4.47\times 10^2\text{ m}$$

$$\vec{F}_0 = -Gm_0\left(\frac{m_1}{\left|\vec{r}_0-\vec{r}_1\right|^3}\left(\vec{r}_0-\vec{r}_1\right) + \frac{m_2}{\left|\vec{r}_0-\vec{r}_2\right|^3}\left(\vec{r}_0-\vec{r}_2\right)\right)$$

$$= -6.67\times 10^{-11}\times 300\left(\frac{100}{500^3}\left(300\hat{i}+400\hat{j}\right) + \frac{200}{\left(4.47\times 10^2\right)^3}\left(-200\hat{i}+400\hat{j}\right)\right)\text{ N}$$

$$= -6.67\times 10^{-11}\times 3\times 10^4\left(\left(\frac{3}{5^3} - \frac{4}{4.47^3}\right)\hat{i} + \left(\frac{4}{5^3} + \frac{8}{4.47^3}\right)\right)\text{ N} = \left(-0.416\hat{i} + 2.43\hat{j}\right)10^{-7}\text{ N}$$

Determining the Mass of Earth: Consider a particle (mass m) located at an altitude h above the surface of earth (mass m_e). Let $\vec{r}$ be the position vector of the particle with respect to the center of earth. The distance between the center of earth and the particle is the sum of radius $\left(R_e\right)$ of earth and the altitude: $\vec{r} = \left(R_e + h\right)\hat{e}_r$ where $\hat{e}_r = \frac{\vec{r}}{r}$ is a unit vector in the direction of the position vector of the particle. Therefore the gravitational force $\left(\vec{F}_e\right)$ exerted by earth on the particle is given as $\vec{F}_e = -G\frac{mm_e}{\left(R_e+h\right)^2}\hat{e}_r$. From Newton's second law, the acceleration $\left(\vec{a}_e\right)$ of the particle due to the gravitational force is the ratio of the gravitational the mass of the particle $\left(\vec{a}_e = \frac{\vec{F}_e}{m}\right)$.

$$\vec{a}_e = -\frac{Gm_e}{\left(R_e+h\right)^2}\hat{e}_r$$

The magnitude of the acceleration of the particle is inversely proportional to the square of the distance between the particle and the center of earth. At the surface of earth, $h = 0$ and the gravitational acceleration on the surface of earth $\left(\vec{g}\right)$ is given as

$$\vec{g} = \vec{a}_e\Big|_{h=0} = -\frac{Gm_e}{R_e^{\,2}}\hat{e}_r$$

Assuming, the radius of earth to be a constant, the magnitude of gravitational acceleration $\left(|g| = \frac{Gm_e}{R_e^{\,2}}\right)$ at the surface of earth is a constant. Since gravitational acceleration and the radius of earth can be measured, the mass of earth can be calculated from this equation $\left(m_e = \frac{|g|R_e^{\,2}}{G}\right)$. With $|g| = 9.8\text{ m/s}^2$ and $R_e = 6.37\times 10^6\text{ m}$, the mass of earth can be calculated to be 5.96×10^{24} kg which is very close to the accepted value of 5.98×10^{24} kg.

14.1 ORBITS DUE TO GRAVITATIONAL FORCE

Consider an object of mass m in orbit around an object of mass M due to gravitational force. If $m \ll M$ the orbit can be considered to be a circular orbit. Thus motion in an orbit can be characterized by the condition that the radius (r) is a constant or $\frac{dr}{dt}=0$. The force responsible for the motion is $\vec{F}=F_r\hat{e}_r+F_\theta=-G\frac{mM}{r^2}\hat{e}_r$ which implies $F_r=-\frac{GmM}{r^2}$ and $F_\theta=0$. The acceleration is $\vec{a}=a_r\hat{e}_r+a_\theta\hat{e}_\theta=\frac{dv_\theta}{dt}\hat{e}_\theta-\frac{v_\theta{}^2}{r}\hat{e}_r$ (where v_θ is the speed of the object along the trajectory), which implies $a_\theta=\frac{dv_\theta}{dt}$ and $a_r=-\frac{v_\theta{}^2}{r}$. Now $F_\theta=ma_\theta=m\frac{dv_\theta}{dt}=0$ implies that the speed of an object in an orbit is a constant. And $F_r=ma_r\Rightarrow -G\frac{mM}{r^2}=-\frac{v_\theta{}^2}{r}$ or with $|v_\theta|=v$,

$$v=\sqrt{\frac{GM}{r}}$$

For an object in a gravitational orbit, the speed of the object is inversely proportional to the square root of the radius of the trajectory.

Example: Consider the motion of earth around the sun. Mass of sun and distance between earth and sun are $M=1.991\times10^{30}$ kg; 1.496×10^{11} m; $v=?$ respectively. (Assume a circular orbit.)

a) Calculate the speed with which earth is revolving around the sun.

Solution:
$M=1.991\times10^{30}$ kg; $r=1.496\times10^{11}$ m; $v=?$

$$v=\sqrt{\frac{GM}{r}}=\sqrt{\frac{6.67\times10^{-11}\times1.991\times10^{30}}{1.496\times10^{11}}}\text{ m/s}=2.97943\times10^{4}\text{ m/s}$$

b) Calculate the time taken for earth to make one complete revolution around earth in days.

Solution:
$T=?$

$$T=\frac{2\pi r}{v}=\frac{2\pi\left(1.496\times10^{11}\right)}{2.97943\times10^{4}}\text{ s}=31548468\text{ s}\left(\frac{\text{min}}{60\text{ s}}\times\frac{\text{hr}}{60\text{ min}}\times\frac{\text{day}}{24\text{ hr}}\right)=365\text{ days}$$

14.2 KEPLER'S LAWS OF PLANETARY MOTION

Kepler's laws are laws that govern motion of planets around the sun. There are three of them.

Kepler's first law states that the planets revolve around the sun in elliptical orbits with the sun on one of the foci of the ellipse. For a number of planets, the elliptical path can be approximated by a circular orbit without the loss of a lot of accuracy.

Kepler's second law states that the radius from the sun to the planet sweeps equal areas in equal intervals of time. This law is a direct consequence of the fact that the angular momentum of the planets is conserved. The angular momentum is conserved for any central force because torque due to a central force is zero (That is, $\vec{\tau} = \vec{r} \times f(r)\frac{\vec{r}}{r} = \frac{f(r)}{r}\vec{r} \times \vec{r} = 0$). It follows that $\vec{L} = \vec{r} \times m\mathrm{v} = m\vec{r} \times \frac{d\vec{r}}{dt} = \frac{m}{dt}(\vec{r} \times d\vec{r}) =$ constant. But $|\vec{r} \times d\vec{r}|$ is equal to the area of the parallelogram formed by the position vector of the planet with respect to the sun and the displacement of the particle in a time interval dt which is twice the area (dA) swept by the position vector in a time interval dt. That is $L = 2m\frac{dA}{dt} =$ constant or $\frac{dA}{dt} =$ constant which is a mathematical statement of Kepler's second law.

Kepler's third law states that the square of the period of a planet is proportional to the cube of the distance between the planet and the sun. This law was deduced by Kepler empirically before the law of gravitation was discovered by Newton. But now it can be shown easily using Newton's law of gravitation. A planet mass m will revolve in an orbit around the sun $(\text{mass } m_s)$ with a uniform speed, when its speed is in such a way that Newton's second law $F = ma$ is satisfied. But for a circular motion with uniform speed $a = \frac{v^2}{r}$; and F is equal to the gravitational force exerted by the sun on the planet. Therefore the equation $\frac{mv^2}{r} = \frac{Gmm_s}{r^2}$ holds. Using $v = \frac{2\pi r}{T}$ (where T is the period), this equation simplifies to

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right) r^3$$

Since $\frac{4\pi^2}{GM_s}$ is a constant, it follows that the square of the period of a planet is proportional to the cube of its distance from the sun. In other words the ratio $\frac{T^2}{r^3}$ is a constant for any planet. For two planets, identified with the subscripts '1' and '2', this proportionality can be represented as

$$\frac{T_1^2}{r_1^3} = \frac{T_2^2}{r_2^3}$$

Example: The distance between the Sun and Jupiter is 7.78×10^{11} m. How long does Jupiter take to make one complete revolution around the sun in earth days? (The distance between Earth and Sun is $r_e = 1.496\times10^{11}$ m.)

Solution: Let's take our 2 planets to be Earth and Jupiter. Let the subscripts 'e' and 'j' represent Earth and Jupiter respectively.

$r_e = 1.496\times10^{11}$ m; $T_e = 365$ days; $r_j = 7.78\times10^{11}$ m; $T_j = ?$

$$\frac{T_e^2}{r_e^3} = \frac{T_j^2}{r_j^3}$$

$$\frac{(365 \text{ day})^2}{(1.496\times10^{11})^3} = \frac{T_j^2}{(7.78\times10^{11})^3}$$

$$T_j^2 = \left(\frac{7.78}{1.496}\right)^3 \cdot 365^2 \text{ day}^2$$

$$T_j = \sqrt{\left(\frac{7.78}{1.496}\right)^3 \times 365^2} \approx 4328.8 \text{ earth days or } 11.86 \text{ earth years}$$

Practice Quiz 14.1

Choose the best answer

1. Which of the following is not true about gravitational force between two objects.
   A) It is an attractive force.
   B) It is inversely proportional to the square of the distance separating them.
   C) It is proportional to the product of their masses.
   D) It is proportional to the sum of their masses.
   E) It is directed along the line joining their centers.

2. By how much would the gravitational force between two objects be multiplied if the mass of one of the objects is multiplied by a factor of *2* and the mass of the other object is multiplied by a factor of *3*?
   A) *6*
   B) *5*
   C) *0.167*
   D) *0.667*
   E) *1.5*

3. Object *1* and Object *2* are located on the x-axis at $x = 300$ m and $x = 400$ m respectively. What is the direction of the gravitational force exerted by object *1* on object *2*?
   A) East
   B) North
   C) West
   D) It cannot be determined
   E) South

4. An object of mass *4000* kg and an object of mass *3000* kg are separated by a distance of *300* m. Calculate the gravitational force exerted by one on the other.
   A) *10.378e-9* N
   B) *16.782e-9* N
   C) *7.569e-9* N
   D) *5.17e-9* N
   E) *8.893e-9* N

5. A satellite of mass *800000* kg is revolving around earth at an altitude of *8000* km above the surface of earth. Earth has a mass of *5.98e24* kg and a radius of *6.38e6* m. Calculate the acceleration with which the satellite is revolving around earth.
   A) *1.929* m/s^2
   B) *3.239* m/s^2
   C) *1.62* m/s^2
   D) *1.133* m/s^2
   E) *1.337* m/s^2

6. Object A of mass *1000* kg is placed *190* m to the left of Object B of mass *1950* kg. Object C of mass *2900* kg is placed *410* m to the right of object B. Calculate the magnitude and direction of the net gravitational force exerted on Object B by objects A and C.
   A) *1.359e-9* N west
   B) *0.938e-9* N east
   C) *1.359e-9* N east
   D) *0.747e-9* N west
   E) *0.938e-9* N west

7. Object A of mass *1500* kg is placed on the origin. Object B of mass *1250* kg is located at y = *160* m on the y-axis. Object C of mass *2100* kg is placed at x = *400* m on the x-axis. Calculate the magnitude of the net gravitational force exerted on Object C by objects A and B.
   A) *2.217e-9* N
   B) *0.773e-9* N
   C) *3.178e-9* N
   D) *4.082e-9* N
   E) *2.671e-9* N

8. Object A of mass *2600* kg is located at y = *-140* m on the y-axis. Object B of mass *1850* kg is located at y = *140* m on the y-axis. Object C of mass *1800* kg is located at x = *410* m on the x-axis. Calculate the net gravitational force exerted on Object C by objects A and B.
   A) (*-499.464e-11* $\boldsymbol{i}$ + *23.386e-11* $\boldsymbol{j}$) N
   B) (*-499.464e-11* $\boldsymbol{i}$ + *15.502e-11* $\boldsymbol{j}$) N
   C) (*-269.368e-11* $\boldsymbol{i}$ + *15.502e-11* $\boldsymbol{j}$) N
   D) (*-269.368e-11* $\boldsymbol{i}$ + *13.75e-11* $\boldsymbol{j}$) N
   E) (*-71.015e-11* $\boldsymbol{i}$ + *13.75e-11* $\boldsymbol{j}$) N

9. Calculate gravitational acceleration due to earth at an altitude of *7700* km. (Radius and mass of earth are *6.37e6* m and *5.98e24* kg respectively).
   A) *3.573* m/s^2
   B) *2.015* m/s^2
   C) *2.878* m/s^2
   D) *2.462* m/s^2
   E) *1.708* m/s^2

10. Which of the following is a correct statement?
   A) According to Kepler's first law, the orbit of the orbit of the planets around the sun is circular.
   B) There is a non-zero net torque acting on a planet as it revolves around the sun.
   C) According to Kepler's third law, the period of a planet is proportional to its radius of revolution.
   D) According to Kepler's second law, as the planet revolves around the sun, the area swept by the radius vector in equal intervals of time varies depending on the location of the planet.
   E) The angular momentum of a planet remains a constant as it revolves around the sun.

11. Use Kepler's law to calculate the time taken by the planet Jupiter to make one complete revolution around the sun in terms of earth years. (Jupiter and earth have radius of revolutions of *7.78e11* m and *1.496e11* m respectively).
    A) *19.689* earth years
    B) *16.89* earth years
    C) *11.86* earth years
    D) *20.987* earth years
    E) *2.404* earth years

14.3 GRAVITATIONAL FIELD

In *field theory,* instead of saying an object exerts force on any object placed at any point in space, it is assumed that the object sets up gravitational field throughout space which exerts force on an object placed at the point. A field is a property of space. It depends only on the coordinates of space and possibly spatial derivatives.

The *gravitational field* due to a certain object at a given point is defined to be the gravitational force per a unit mass exerted on a small mass placed at the given point. Consider the gravitational field due to an object of mass M whose position vector is $\vec{r}\,'$ at a point whose position vector is $\vec{r}$. Let's assume a small object of mass m is placed at the point, then the gravitational force exerted on it is $\vec{F} = -G\frac{mM}{|\vec{r}-\vec{r}\,'|^3}(\vec{r}-\vec{r}\,')$. The gravitational field $(\vec{g})$ at this point is defined to be the ratio of this force acting on it to its mass m $\left(\vec{g} = \frac{\vec{F}}{m}\right)$.

$$\vec{g} = -G\frac{M}{|\vec{r}-\vec{r}\,'|^3}(\vec{r}-\vec{r}\,')$$

If the origin is chosen to be at the center of the object producing the field, $\vec{r}\,' = 0$ and this expression simplifies to

$$\vec{g} = -\frac{GM}{r^3}\vec{r} = -\frac{GM}{r^2}\hat{e}_r$$

Where $\hat{e}_r = \frac{\vec{r}}{r}$ is a unit vector in the direction of the position vector of the point with respect to the center of the object. The unit of measurement for gravitational field is m/s^2 ; which is the unit of acceleration. The gravitational field at a certain point is equal to the acceleration due to gravitational force of a particle placed at the point. If the gravitational field at a certain point is $\vec{g}$, then the gravitational force experienced on a particle of mass m placed at the point is $\vec{F} = m\vec{g}$.

Example: Calculate the gravitational field due to an object of mass 10^6 kg at a point 1000 m above it.

Solution: Using a coordinate system where the y-axis lies along the line joining the object and the point and where the origin is located at the object, $\vec{r} = 1000\hat{j}$ m.

$M = 10^6$ kg; $\vec{r} = 1000\hat{j}$; $r = 1000$ m; $\vec{g} = ?$

$$\vec{g} = -\frac{GM}{r^3}\vec{r} = -\frac{6.67\times10^{-11}\times10^6}{1000^3}\left(1000\hat{j}\right)\text{ m/s}^2 = -6.67\times10^{-14}\,\hat{j}\text{ m/s}^2$$

Superposition Principle: If a point is in the vicinity of a number of particles, then the gravitational field at the point is the vector sum of all the gravitational fields due to the objects in its vicinity. If objects of masses $M_1, M_2, M_3, \ldots$ of position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots$ respectively are in the vicinity of a point whose position vector is $\vec{r}$, then the net gravitational field at the point is given as

$$\vec{g} = -G\left(\frac{M_1}{\left|\vec{r}-\vec{r}_1\right|^3}\left(\vec{r}-\vec{r}_1\right) + \frac{M_2}{\left|\vec{r}-\vec{r}_2\right|^3}\left(\vec{r}-\vec{r}_2\right) + \frac{M_3}{\left|\vec{r}-\vec{r}_3\right|^3}\left(\vec{r}-\vec{r}_3\right) + \ldots\right) = -G\sum_i \frac{M_i}{\left|\vec{r}-\vec{r}_i\right|^3}\left(\vec{r}-\vec{r}_i\right)$$

Example: An object of mass 2000 kg is placed 4000 m to the right of an object whose mass is 1000 kg. Calculate the net gravitational field at a point located 1000 m to the right of the 1000 kg object.

Solution: Let's use a coordinate system where the x-axis lies along the line joining the two objects and where the origin is located at the 1000 kg object.

$$M_1 = 1000 \text{ kg};\ \vec{r}_1 = 0;\ M_2 = 2000 \text{ kg};\ \vec{r}_2 = 4000\hat{i} \text{ m};\ \vec{r} = 1000\hat{i} \text{ m};\ \vec{g} = ?$$

$$\vec{r} - \vec{r}_1 = 1000\hat{i} \text{ m} - 0 = 1000\hat{i} \text{ m}$$

$$|\vec{r} - \vec{r}_1| = 1000 \text{ m}$$

$$\vec{r} - \vec{r}_2 = (1000 - 4000)\hat{i} \text{ m} = -3000\hat{i} \text{ m}$$

$$|\vec{r} - \vec{r}_2| = 3000 \text{ m}$$

$$\vec{g} = -G\left(\frac{M_1}{|\vec{r} - \vec{r}_1|^3}(\vec{r} - \vec{r}_1) + \frac{M_2}{|\vec{r} - \vec{r}_2|^3}(\vec{r} - \vec{r}_2)\right)$$

$$\vec{g} = -6.67 \times 10^{-11}\left(\frac{1000}{1000^3}(1000\hat{i}) + \frac{2000}{3000^3}(-3000\hat{i})\right) \text{ N} = -5.2 \times 10^{-14}\hat{i} \text{ N/kg}$$

Example: Two objects of masses 1000 kg and 2000 kg are located on the points $(0, 3000)$ m and $(0, -3000)$ m respectively.

a) Calculate the net gravitational field at the point $(4000, 0)$ m.

Solution:

$$M_1 = 1000 \text{ kg};\ \vec{r}_1 = (0, 3000) \text{ m} = 3000\hat{j} \text{ m};\ M_2 = 2000 \text{ kg};\ \vec{r}_2 = (0, -3000) \text{ m} = -3000\hat{j} \text{ m};$$

$$\vec{r} = (4000, 0) \text{ m} = 4000\hat{i} \text{ m};\ \vec{g} = ?$$

$$\vec{r} - \vec{r}_1 = (4000\hat{i} - 3000\hat{j}) \text{ m}$$

$$|\vec{r} - \vec{r}_1| = \sqrt{4000^2 + (-3000)^2} \text{ m} = 5000 \text{ m}$$

$$\vec{r} - \vec{r}_2 = (4000\hat{i} + 3000\hat{j}) \text{ m}$$

$$|\vec{r} - \vec{r}_2| = \sqrt{4000^2 + (3000)^2} \text{ m} = 5000 \text{ m}$$

$$\vec{g} = -G\left(\frac{M_1}{|\vec{r}-\vec{r}_1|^3}(\vec{r}-\vec{r}_1) + \frac{M_2}{|\vec{r}-\vec{r}_2|^3}(\vec{r}-\vec{r}_2)\right)$$

$$= -6.67\times10^{-11}\left(\frac{1000}{5000^3}\left(4000\hat{i}-3000\hat{j}\right) + \frac{2000}{5000^3}\left(4000\hat{i}+3000\hat{j}\right)\right) \text{ N/kg}$$

$$= \left[-6.4\times10^{-15} - 1.6\times10^{-15}\hat{j}\right] \text{ N/kg}$$

b) Calculate the gravitational force exerted on an object of mass 100 kg placed at point.

Solution:
$m = 100$ kg; $\vec{F} = ?$

$$\vec{F} = m\vec{g} = 100\left(-6.4\times10^{-15} - 1.6\times10^{-15}\hat{j}\right) \text{ N} = \left(-6.4\times10^{-13} - 1.6\times10^{-13}\hat{j}\right) \text{ N}$$

14.4 GRAVITATIONAL POTENTIAL ENERGY

The gravitational potential energy stored by two objects of masses m and M separated by a distance r is defined to be equal to the amount of work needed to be done by an external force to bring one of the objects, say the object of mass m, from infinity to a separation distance r from the other object. As the object is displaced from infinity to its location, the forces acting on it are the gravitational force $\left(\vec{F}_g = -G\frac{mM}{r^2}\hat{e}_r\right)$ exerted by M and the external force. Work needed by the external force means, the minimum amount needed without accelerating the object. Therefore the net work done on the object should be zero; that is $w_{ext} + w_g = 0$ where w_{ext} and w_g represent the work done by the external force and the gravitational force respectively. Therefore $w_{ext} = -w_g = -\int_{\infty}^{r}\vec{F}_g\cdot d\vec{r} = GmM\int_{\infty}^{r}\frac{\hat{e}_r\cdot d\vec{r}}{r^2}$. Since gravitational force is conservative, this integral is path independent. Assuming the object is displaced in the direction of $\hat{e}_r$, $d\vec{r} = dr\hat{e}_r$ and $w_{ext} = GmM\int_{\infty}^{r}\frac{dr}{r^2} = -\frac{GmM}{r}$ which is equal to the gravitational potential energy (U) stored by the two objects.

$$U = -\frac{GmM}{r}$$

The potential energy stored by two objects is inversely proportional to the distance separating them.

Example: Calculate the gravitational potential energy stored by earth-satellite system when a satellite of mass 100,000 kg is revolving at an altitude of 1000 km (Mass of earth $= 5.98\times10^{24}$ kg; radius of earth $= 6.37\times10^{6}$m).

Solution: The distance between earth and the satellite is equal to the sum of earth's radius and the altitude.

$$M = 5.98\times10^{24}\ \text{kg};\ m = 10^{5}\ \text{kg};\ h = 10^{6}\ \text{m};\ R_e = 6.37\times10^{6}\ \text{m};\ U = ?$$

$$r = R_e + h = \left(6.37\times10^{6} + 10^{6}\right)\ \text{m} = 7.37\times10^{6}\ \text{m}$$

$$U = -\frac{GmM}{r} = -\frac{6.67\times10^{-11}\times10^{5}\times5.98\times10^{24}}{7.37\times10^{6}}\ \text{J} = -5.4\times10^{12}\ \text{J}$$

Example: Calculate the work done by gravitational force when an object of mass 1000 kg falls to the surface of earth from an altitude of 1000 m.

Solution: Since gravitational force is conservative, the work done by gravitational force is equal to the negative change of gravitational potential energy.

$h = 1000 \text{ m}; \; w_g = ?$

$$r_i = R_e + h = \left(6.37\times10^6 + 10^3\right) \text{ m} = 6.371\times10^6 \text{ m}$$

$$r_f = R_e = 6.37\times10^6 \text{ m}$$

$$w_g = -\Delta U = -\left(U_f - U_i\right) = -\left[\left(-\frac{GMm}{r_f}\right) - \left(-\frac{GMm}{r_i}\right)\right] = GMm\left[\frac{1}{r_f} - \frac{1}{r_i}\right]$$

$$= \left(6.67\times10^{-11}\right)\left(10^3\right)\left(5.98\times10^{24}\right)\left[\frac{1}{6.37\times10^6} - \frac{1}{6.371\times10^6}\right] \text{ J} = 9828 \text{ J}$$

14.5 CONSERVATION OF MECHANICAL ENERGY

Since gravitational force is conservative, mechanical energy is conserved if only gravitational forces are involved. Consider a system involving two objects of masses M and m, where M is much bigger than m. If $M >> m$, then M can essentially be treated as a stationary object. The mechanical energy of the system can be approximated as the sum of the potential energy of the system, and the kinetic energy of the smaller object (neglecting the kinetic energy of the bigger object); That is

$$ME = \frac{1}{2}mv^2 - \frac{GmM}{r}$$

And the principle of conservation of mechanical energy for gravitational forces where $M \gg m$ can be written as

$$-\frac{GMm}{r_i} + \frac{1}{2}mv_i^2 = -\frac{GMm}{r_f} + \frac{1}{2}mv_f^2$$

Where r_i $\left(r_f\right)$ is the initial (final) separation between the objects and v_i $\left(v_f\right)$ is the initial (final) speed of the smaller object.

Example: An object of mass 100,000 kg is released from an altitude of 1000 m. Assuming no air resistance, calculate the speed of the object by the time it reaches the surface of earth.

Solution:
$h = 1000$ m; $m = 10^5$ kg; $M = 5.98\times10^{24}$ kg; $v_i = 0$ (released from rest); $v_f = ?$

$$r_i = R_e + 100 \text{ m} = 6.37\times10^6 \text{ m} + 10^3 \text{ m} = 6.371\times10^6 \text{ m}$$

$$r_f = R_e = 6.37\times10^6 \text{ m}$$

$$-\frac{GMm}{r_i} + \frac{1}{2}mv_i^2 = -\frac{GMm}{r_f} + \frac{1}{2}mv_f^2$$

$$v_f^2 = 2GM\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

$$v_f = \sqrt{2GM\left(\frac{1}{r_f} - \frac{1}{r_i}\right)} = \sqrt{2\left(6.67\times10^{-11}\right)\left(5.98\times10^{24}\right)\left[\frac{1}{6.37\times10^6} - \frac{1}{6.371\times10^6}\right]} = 140 \text{ m/s}$$

14.6 KINETIC AND MECHANICAL ENERGY OF OBJECTS IN ORBIT

Consider an object of mass m revolving around a massive object of mass M ($M >> m$) at a radius of r with a speed of v. Its kinetic energy is $KE = \frac{1}{2}mv^2$. v^2 can be expressed in terms of r by using Newton's second law: $\frac{mv^2}{r} = \frac{GmM}{r^2} \Rightarrow v^2 = \frac{GM}{r}$ Therefore for an object of mass m revolving in an orbit of radius r around a massive object of mass M is given as

$$KE = \frac{1}{2}\frac{GMm}{r}$$

And the mechanical energy of an object in an orbit, which is the sum of its potential and kinetic energy is given as $ME = -\frac{GMm}{r} + \frac{1}{2}\frac{GMm}{r}$ or

$$ME = -\frac{1}{2}\frac{GMm}{r}$$

We see that, for an object in an orbit, there are very simple relationships between the potential energy, kinetic energy and mechanical energy.

$$KE = -\frac{1}{2}U$$
$$ME = \frac{1}{2}U$$

The mechanical energy is equal to the negative of the kinetic energy.

Example: A satellite of mass 100,000 kg is revolving at an altitude of 1000 m. How much energy is needed to put it in orbit at an altitude of 2000 m?

Solution: Let the work done by the external force be denoted as w_{ext} The forces acting on the object are gravitational force and the external force for changing the orbit. Therefore the net work done is equal to the sum of the work done $\left(w_{ext}\right)$ by the external force and the work done $\left(w_g\right)$ by gravitational force which in turn is equal to the change in kinetic energy.

$h_i = 1000$ m; $h_f = 2000$ m; $m = 10^5$ kg; $M = 5.98\times10^{24}$ kg; $w_{ext} = ?$

$$w_{net} = w_{ext} + w_g = \Delta KE$$

$$w_g = -\Delta U$$

$$w_{net} = w_{ext} - \Delta U = \Delta KE \Rightarrow w_{ext} = \Delta U + \Delta KE = \Delta ME = \frac{GMm}{2}\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

$$r_i = R_e + h_i = \left(6.37\times10^6 + 10^3\right)\ \text{m} = 6.371\times10^6\ \text{m}$$

$$r_f = R_e + h_f = \left(6.37\times10^6 + 2\times10^3\right)\ \text{m} = 6.372\times10^6\ \text{m}$$

$$w_{ext} = \frac{1}{2}\left(6.67\times10^{-11}\right)\left(10^5\right)\left(5.98\times10^{24}\right)\left(10^{-6}\right)\left(\frac{1}{6.37} - \frac{1}{6.372}\right)\ \text{J} = 4.9\times10^8\ \text{J}$$

14.7 ESCAPE VELOCITY

Escape velocity (v_{esc}) is the minimum velocity for which an object will never return to earth when propelled upwards from the surface of earth. The minimum velocity is the velocity for which the final speed is zero when the object is free from gravitational field. The object is free from gravitational field of earth, when it is separated by an infinite distance from earth. An expression for the escape velocity may be obtained from the principle of conservation of mechanical energy with $r_i = R_e;\ v_i = v_{esc} = ?;\ r_f = \infty;\ v_f = 0$. $-\frac{Gm_e m}{r_i} + \frac{1}{2} m v_{esc}^2 = -\frac{Gm_e m}{r_f} + \frac{1}{2} m v_f^2$ where m_e is the mass of earth implies the following expression for the escape velocity.

$$v_{esc} = \sqrt{\frac{2Gm_e}{R_e}}$$

We see that the escape velocity does not depend on the mass of the object being propelled. It is a constant for all objects and its value is $v_{esc} = \sqrt{\frac{2Gm_E}{R_E}} = \sqrt{\frac{2(6.67\times10^{-11})(5.98\times10^{24})}{6.37\times10^{6}}}\ \text{m/s} = 1.12\times10^{4}\ \text{m/s}$.

Example: Calculate the energy needed to propel an object of mass 2×10^3 kg for which the object will never return to the surface of earth (assume no air resistance).

Solution:
$m = 2\times10^3$ kg; $KE = ?$

$$KE = \frac{1}{2} m v_{esc}^2 = \frac{1}{2}\left(2\times10^{-3}\right)(1.12\times10^4)^2\ \text{J} = 1.23\times10^6\ \text{J}$$

Practice Quiz 14.2

Choose the best answer

1. Calculate the gravitational field due to an object of mass *9.4e6* kg at a point located *6.1e3* m away to the left of the object.
   A) *-26.422e-12* N / kg ***i***
   B) *16.85e-12* N / kg ***i***
   C) *-9.194e-12* N / kg ***i***
   D) *-16.85e-12* N / kg ***i***
   E) *9.194e-12* N / kg ***i***

2. An object of mass *8.4e3* kg is located at the origin. An object of mass *1.5e3* kg is located on the x-axis at x = *1000* m. Calculate the gravitational field at a point located on the x-axis at x = *3.6e2* m.
   A) *-46.133e-14* N / kg ***i***
   B) *-735.145e-14* N / kg ***i***
   C) *-296.356e-14* N / kg ***i***
   D) *-186.848e-14* N / kg ***i***
   E) *-407.889e-14* N / kg ***i***

3. An object of mass *7.4e3* kg is located at the origin. An object of mass *5.1e4* kg is located on the x-axis at x = *1000* m. The gravitational field will be zero at x =
   A) *339.39* m
   B) *275.844* m
   C) *82.923* m
   D) *370.318* m
   E) *115.473* m

4. An object of mass *9.4e3* kg is located on the y-axis at y = *8.2e2* m. An object of mass *3.2e3* kg is located on the y-axis at y = *-8.2e2* m. Calculate the magnitude of the gravitational field at a point on the x-axis at x = *8.2e2* m
   A) *75.019e-14* N / kg
   B) *33.301e-14* N / kg
   C) *49.25e-14* N / kg
   D) *18.802e-14* N / kg
   E) *60.735e-14* N / kg

5. Calculate the mechanical energy of an object of mass *2.4e5* kg moving with a speed of *0.9e3* m / s at an altitude of *2.8e6* m. (The mass and the radius of earth are *5.98e24* kg and *6.37e6* m respectively).
   A) *-141.709e11* J
   B) *-177.826e11* J
   C) *-43.641e11* J
   D) *-103.42e11* J
   E) *-26.609e11* J

6. Calculate the work done by gravitational force as an object of mass *7.4e5* kg falls to the surface of earth from an altitude of *4.5e6* m. (The mass and the radius of earth are *5.98e24* kg and *6.37e6* m respectively).
   A) *72.266e11* J
   B) *191.824e11* J
   C) *51.655e11* J
   D) *282.871e11* J
   E) *309.295e11* J

7. An object of mass *5.1e5* kg is fired upwards from the ground with a speed of *9.5e3* m / s. Assuming no air resistance, determine how high it would rise. (The mass and the radius of earth are *5.98e24* kg and *6.37e6* m respectively).
   A) *5.121e6* m
   B) *16.434e6* m
   C) *28.664e6* m
   D) *12.596e6* m
   E) *20.5e6* m

8. Calculate the potential energy of a satellite revolving in an orbit around earth with a kinetic energy of *3.2e12* J.
   A) *-1.6e12* J
   B) *-6.4e12* J
   C) *3.2e12* J
   D) *6.4e12* J
   E) *-3.2e12* J

9. Calculate the amount of energy needed (by an external source) to change the altitude of the orbit of a satellite of mass *6.7e5* kg from *7.7e6* m to *1.5e7* m. (The mass and the radius of earth are *5.98e24* kg and *6.37e6* m respectively).
   A) *324.411e11* J
   B) *73.156e11* J
   C) *476.164e11* J
   D) *543.328e11* J
   E) *268.631e11* J

10. A heavenly object has a mass of *5.3e24* kg and a radius of *2.4e6* m. Calculate the minimum energy needed to propel an object of mass *100* kg from its surface if the object is to never return to its surface (neglect any influence from other objects).
   A) *194.761e8* J
   B) *218.713e8* J
   C) *147.296e8* J
   D) *60.504e8* J
   E) *103.88e8* J

15 OSCILLATORY MOTION

Your goal for this chapter is to learn the nature of objects undergoing oscillatory motion.

An oscillatory motion is a back and forth motion as a function of time. Often, an oscillatory motion is also a periodic motion. A periodic motion is a motion in which a certain pattern repeats itself again and again. The time taken by a periodic motion to make one complete pattern is called period (T). And the number of patterns executed per second is called frequency (f). Frequency and period are inverses of each other.

$$f = \frac{1}{T}$$

The unit of frequency, which is 1/second, is defined to be Hertz, abbreviated as Hz. A very common type of oscillatory motion is a motion where the displacement of the particle varies like a sine or a cosine as a function of time. Such kind of motion is called simple harmonic motion.

15.1 SIMPLE HARMONIC MOTION

A *simple harmonic motion* is a motion where the force acting on the particle is proportional and opposite in direction to the displacement of the particle. For one dimensional motion, this may be expressed mathematically as

$$F_x = -cx$$

Where F_x is the force acting on the particle, x is the displacement of the particle with respect to its equilibrium position, and c is a constant of proportionality called *force constant*. The unit of measurement for c is Newton/meter (N/m). Since from Newton's second law $F_x = ma_x$, the acceleration of the particle is also proportional to the displacement of the particle: $a_x = -\frac{c}{m}x$. And since $a_x = \frac{d^2x}{dt^2}$, it follows that for a simple harmonic motion the differential equation $\frac{d^2x}{dt^2} + \frac{c}{m}x = 0$ applies. With $\sqrt{\frac{c}{m}} = \omega_o$, the differential equation satisfied by the displacement of a particle executing harmonic motion may be written as

$$\frac{d^2x}{dt^2} + \omega_o^2 x = 0$$

This is a second order differential equation. The general solution of a 2nd order differential equation is two dimensional, which means all of the solutions of the equation can be expressed in terms of two linearly independent solutions of the equation (two functions are said to be linearly independent if they are not proportional to each other). By direct substitution, it can be shown that the functions $x(t)=\cos(\omega_0 t)$ and $x(t)=\sin(\omega_0 t)$ are solutions of this equation. Since $\cos\omega_o t$ and $\sin\omega_o t$ are linearly independent, the general solution of this equation can be written as a linear combination of these functions:

$$x(t)=c_1\cos\omega_o t+c_2\sin\omega_o t$$

Where c_1 and c_2 are arbitrary constants to be determined by two initial conditions. These two terms can be combined into one term by transforming the constants c_1 and c_2 into the constants A and ϕ using the transformation equations $c_1=A\cos\phi$ and $c_1=A\sin\phi$. Therefore, in terms of these constants, the expression for the general solution becomes $x(t)=A\cos\phi\cos\omega_o t+A\sin\phi\sin\omega_o t$. These two terms can be combined into one term using the formula for the cosine of the difference of two angles $\left(\cos(a-b)=\cos a\cos b+\sin a\sin b\right)$ as

$$x(t)=A\cos(\omega_o t-\phi)$$

The constant A represents the maximum value of the displacement of the particle and is called the amplitude of the motion. The constant $\omega_o = \sqrt{c/m}$ determines how fast the particle oscillates back and forth and is called angular frequency of the motion. Its unit is rad/second. When $t = T_0$ (the period), $\omega_0 T_0 = 2\pi$ (which is the period of a cosine function) or $\omega_0 = \frac{2\pi}{T_0} = 2\pi f_0$. Therefore, for a particle undergoing a harmonic motion, the following relationships between the oscillation properties and the force constant hold.

$$\omega_o = \sqrt{\frac{c}{m}},\ f_0 = \frac{1}{2\pi}\sqrt{\frac{c}{m}},\ \text{and } T_0 = 2\pi\sqrt{\frac{m}{c}}$$

The constant ϕ is called the phase angle of the motion, and its effect is to shift the graph of the cosine function to the right (if positive) or to the left (if negative). The constants A and ϕ can be determined from initial conditions. If $x|_{t=0} = x_o$ then $x_0 = A\cos(\phi)$ If $\left.\frac{dx}{dt}\right|_{t=0} = v_0$ then $v_0 = A\omega_0 \sin(\phi)$ An expression for A can be obtained by squaring the expressions for x_0 and v_0 and adding. An expression for ϕ can be obtained dividing the expression for v_0 by the expression for x_0.

$$A = \sqrt{x_o^2 + \left(\frac{v_o}{\omega_o}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{v_o}{\omega_o x_o}\right)$$

Example: By direct substitution show that $x(t) = A\cos(\omega_o t - \phi)$ is the solution of the equation of a harmonic motion $\frac{d^2x}{dt^2} + \omega_o x = 0$.

Solution:

$$\frac{d^2x}{dt^2} + \omega_o^2 x = \frac{d^2}{dt^2}\left[A\cos(\omega_o t - \phi)\right] + \omega_o^2 A\cos(\omega_o t - \phi)$$
$$= \frac{d}{dt}\left[-A\omega_o \sin(\omega_o t - \phi)\right] + \omega_o^2 A\cos(\omega_o t - \phi)$$
$$= -A\omega_o^2 \cos(\omega_o t - \phi) + A\omega_o^2 \cos(\omega_o t - \phi) = 0$$

Example: Give expressions for the velocity, acceleration and force acting as a function of time for a particle undergoing a simple harmonic motion.

Solution:
$v(t) = ?;\ a(t) = ?;\ F(t) = ?$

$$v = \frac{dx}{dt} = \frac{d}{dt}\left[A\cos\left(\omega_o t - \phi\right)\right] = -A\omega_o \sin\left(\omega_o t - \phi\right)$$
$$a = \frac{dv}{dt} = \frac{d}{dt}\left[-A\omega_o \sin\left(\omega_o t - \phi\right)\right] = -A\omega_o^2 \cos\left(\omega_o t - \phi\right)$$
$$F = ma = -m\omega_o^2 A\cos\left(\omega_o t - \phi\right)$$

15.2 ENERGY OF A HARMONIC OSCILLATOR

The force acting on a harmonic oscillator is conservative. Therefore the mechanical energy of a harmonic oscillator is expected to be conserved. An expression for this conserved mechanical energy can be obtained by adding the kinetic and potential energy of a harmonic oscillator. Assuming the potential energy of the spring to be zero at the relaxed position $(x = 0)$, the potential energy is given as $U_e(x) = -\int_0^x F_{x'}dx' = -\int_0^x -cx'dx' = \frac{1}{2}cx^2$. Using the expressions for the displacement and velocity as a function of time obtained in the preceding example, the mechanical energy of a particle undergoing a harmonic motion is given as $ME = KE + U = \frac{1}{2}mv^2 + \frac{1}{2}cx^2 = \frac{1}{2}m\left(-A\omega_o \sin\left(\omega_o t - \phi\right)\right)^2 + \frac{1}{2}c\left(A\cos\left(\omega_o t - \phi\right)\right)^2$ which simplifies to the following expression for the mechanical energy.

$$ME = \frac{1}{2}m\omega_o^2 A^2 = \frac{1}{2}cA^2$$

The mechanical energy of a harmonic oscillator is independent of time. It depends only on the force constant (c) and amplitude (A) only, which means mechanical energy is conserved as expected. With this expression of the mechanical energy, an expression of the velocity of the particle as a function of displacement can be obtained. That is $ME = \frac{1}{2}mv^2 + \frac{1}{2}m\omega_0{}^2x^2 = \frac{1}{2}m\omega_0{}^2A^2$ implies that

$$v = \pm\omega_o\sqrt{A^2 - x^2}$$

The velocity is positive (negative) when the particle moves to the right (left).

15.3 AN OBJECT ATTACHED TO A SPRING

The dynamics of a spring is governed by Hook's law. Hook's law states that the force due to a spring is proportional and opposite to displacement. This may be mathematically expressed as

$$F_s = -kx$$

The constant of proportionality is called Hook's constant and is constant for a given spring but may be different for different springs. Its unit of measurEment is N/m. Since the force acting on an object attached to a spring is proportional but opposite to displacement, all the equations of a harmonic oscillator apply with $c = k : \omega_o = \sqrt{\frac{k}{m}} = 2\pi f_0 = \frac{2\pi}{T_0}$, $ME = \frac{1}{2} m\omega_o^2 A^2 = \frac{1}{2} kA^2$, and so on.

Example: A certain spring extends by 2 cm when an object of mas 2 kg hands from it. By how much will it extend when an object of mass 7 kg hangs from it?

Solution: Since the force and the displacement are proportional, the ratio of force to displacement is a constant. Since the object is in equilibrium hanging, the force due to the spring is equal to the weight of the object $\left(F_1 = m_1 |g| \text{ and } F_2 = m_2 |g|\ \right)$.

$$m_1 = 2\text{ kg};\ x_1 = 0.02\text{ cm};\ m_2 = 7\text{ kg};\ x_2 = ?$$

$$\frac{m_1 |g|}{x_1} = \frac{m_2 |g|}{x_2}$$

$$x_2 = \frac{m_2}{m_1} x_1 = \frac{7}{2} \times 0.02\text{ m} = 0.07\text{ m}$$

Example: An object of mass 2 kg is attached to a spring of Hook's constant 200 N/m on a horizontal frictionless surface. Then it is extended by 10 cm and then let free to oscillate.

a) How long will it take to make one complete oscillation?

Solution:
$k = 200$ N/m; $m = 2$ kg; $T_0 = ?$

$$T_0 = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2}{200}}\ \text{s} = \frac{\pi}{5}\ \text{s}$$

b) How many oscillations does it execute per second?

Solution:
$f_0 = ?$

$$f_0 = \frac{1}{T_0} = \frac{5}{\pi}\ \text{Hz}$$

c) Determine the amplitude and the phase angle of the motion and then give an expression for the displacement as a function of time.

Solution:
$x_0 = 0.1$ m; $v_0 = 0$ (released from rest); $A = ?$; $\phi = ?$; $x(t) = ?$

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{2}}\ \text{rad/s} = 10\ \text{rad/s}$$

$$A = \sqrt{x_o^2 + \left(\frac{v_o}{\omega_o}\right)^2} = x_0 = 0.1\ \text{m}$$

$$\phi = \tan^{-1}\left(\frac{v_o}{\omega_o x_o}\right) = \tan^{-1}(0) = 0$$

$$x(t) = A\cos(\omega_o t - \phi) = 0.1\cos(10t)\ \text{m}$$

d) Calculate its displacement, velocity, acceleration and force acting on it after $\frac{\pi}{10}$ seconds.

Solution:
$x\big|_{t=\frac{\pi}{10}\,\text{s}} = ?;\ v\big|_{t=\frac{\pi}{10}\,\text{s}} = ?;\ a\big|_{t=\frac{\pi}{10}\,\text{s}} = ?;\ F\big|_{t=\frac{\pi}{10}\,\text{s}} = ?$

$$x(t) = 0.1\cos(10t)\ \text{m}$$

$$x\big|_{t=\frac{\pi}{10}\text{ s}} = 0.1\cos\left(10\times\frac{\pi}{10}\right)\text{ m} = -0.1\text{ m}$$

$$v(t) = \frac{dx}{dt} = -\sin(10t)\text{ m/s}$$

$$v\big|_{t=\frac{\pi}{10}\text{ s}} = -\sin\left(10\times\frac{\pi}{10}\right) = 0$$

$$a(t) = \frac{dv}{dt} = -10\cos(10t)\text{ m/s}^2$$

e) Calculate its velocity by the time it is extended by 0.5 cm.

Solution:

$x = 0.05$ m; $v = ?$

$$v = \pm\omega_o\sqrt{A^2 - x^2} = \pm 10\sqrt{0.1^2 - 0.05^2}\text{ m/s} = \pm 0.866\text{ m/s}$$

f) Calculate the mechanical energy of the object.

Solution:

$ME = ?$

$$ME = \frac{1}{2}kA^2 = \frac{1}{2}(200)(0.1)^2\text{ J} = 1\text{ J}$$

Practice Quiz 15.1

Choose the best answer

1. The unit of measurement for frequency is
   A) second
   B) meter / second
   C) radian / second ^{2}
   D) radian / second
   E) Hertz

2. Which of the following is not a correct statement.
   A) The x-component of an object undergoing a uniform circular motion is harmonic.
   B) All periodic motions are harmonic.
   C) All harmonic motions are periodic.
   D) The motion of a pendulum is generally not harmonic.
   E) The motion of an object attached to a spring is harmonic.

3. A certain harmonic motion varies with time according to the equation $x = 30$ cm $\cos(60t)$. How long does it take to make one complete cycle?
   A) *94.248* s
   B) *0.105* s
   C) *60* s
   D) *0.017* s
   E) *30* s

4. A spring extends by *0.3* m when an object of mass *14* kg hangs from it. Calculate the mass of a hanging object that will extend it by *0.55* m.
   A) *48.626* kg
   B) *39.845* kg
   C) *34.17* kg
   D) *25.667* kg
   E) *11.616* kg

5. The displacement of the motion of a certain object of mass *1.4* kg undergoing a simple harmonic motion satisfies the equation $d^2x/dt^2 + 78.5x = 0$. Calculate the force constant of the motion.
   A) *61.984* N / m
   B) *140.199* N / m
   C) *191.744* N / m
   D) *109.9* N / m
   E) *32.523* N / m

6. The displacement of a certain object of mass *2.1* kg undergoing a simple harmonic motion varies with time according to the equation $x = 0.81 \cos(53.9t - 2.65)$. Calculate its mechanical energy after *9* seconds.
   A) *551.974* J
   B) *278.985* J
   C) *1357.456* J
   D) *2474.564* J
   E) *2001.414* J

7. An object of mass *2.4* kg is attached to a spring of Hook's constant *250* N / m on a friction less horizontal surface. It is extended by *0.05* m and then let free to oscillate. How long does it take to make one complete oscillation?
   A) *0.902* s
   B) *0.616* s
   C) *0.478* s
   D) *0.339* s
   E) *0.784* s

8. An object is attached to a spring of Hook's constant *150* N / m on a friction less horizontal surface. It is extended by *0.05* m and then let free to oscillate. If it makes *10* cycles per second, calculate the mass of the object.
   A) *0.061* kg
   B) *0.026* kg
   C) *0.038* kg
   D) *0.043* kg
   E) *0.007* kg

9. Calculate the elastic potential energy stored by a spring of Hook's constant *25* N / m when compressed by *0.19* m.
   A) *0.451* J
   B) *0.724* J
   C) *0* J
   D) *0.254* J
   E) *0.795* J

10. An object of mass *0.2* kg is attached to a spring of Hooks constant *100* N / m on a friction less horizontal surface. Then it is extended by *0.1* m and then let free to oscillate. By the time it is compressed by *0.01* m, calculate the force exerted by the spring on the object.
   A) *1.825* N
   B) *0.782* N
   C) *0.189* N
   D) *1* N
   E) *0.569* N

11. A spring of Hook's constant *6.6* N / m is resting on a frictionless table with the fixed end being the left end. An object of mass *0.9* kg is attached to the spring and compressed to the left by *0.63* m. Then it is let free to oscillate. Give an equation for its displacement as function of time.
   A) $0.63 \cos(5.002t - \pi/2)$
   B) $0.63 \cos(5.002t)$
   C) $0.63 \cos(1.809t + \pi)$
   D) $0.63 \cos(2.708t)$
   E) $0.63 \cos(2.708t - \pi)$

15.4 A SIMPLE PENDULUM

As will be shown shortly, the motion of a pendulum is harmonic only approximately for small angular displacements. Otherwise, generally, the motion of a pendulum is not harmonic.

Consider a pendulum of length ℓ displaced by an angle θ from its vertical position. The forces acting on the pendulum are its weight $(\vec{w})$ and the tension in the string. The force due to the string does not contribute to the tangential acceleration because it is always perpendicular to the trajectory. The weight makes an angle of θ with the radially outward unit vector $(\hat{e}_r)$ in a clockwise direction (that is the angle formed is $-\theta$). The weight force can be expressed in terms of the radially outward unit vector and the tangential unit vector $(\hat{e}_\theta)$ as $\vec{w} = m|g|\cos(-\theta)\hat{e}_r + m|g|\sin(-\theta)\hat{e}_\theta$. Therefore the tangential force (F_θ) responsible for the tangential acceleration of the pendulum is given as $F_\theta = -m|g|\sin(\theta)$. For small angles (less than 15°) $\sin(\theta)$ is approximately equal to θ which in turn is equal to (when measured in radians) the ratio of the arc-length (s) to the length of the pendulum $\left(\theta = \frac{s}{\ell}\right)$. Thus, for small angles, the tangential force may be written as $F_s \approx -\frac{m|g|}{\ell}s$ which shows that the tangential force is proportional and opposite to the linear displacement with a constant of proportionality $\frac{m|g|}{\ell}$. It follows that, for small angular displacement, the motion of a pendulum is approximately simple harmonic a force constant of $c = \frac{m|g|}{\ell}$ Therefore for a pendulum displaced by a small angle, the angular frequency $w_0 = \sqrt{\frac{c}{m}}$ is given as

$$w_0 = \sqrt{\frac{|g|}{\ell}} = 2\pi f_0 = \frac{2\pi}{T_0}$$

Example: A pendulum of length 0.098 m is displaced by a small angle and let free to oscillate.

a) How long will it take to make on complete oscillation.

Solution:
$\ell = 0.098$ m; $T_0 = ?$

$$T = 2\pi\sqrt{\frac{l}{|g|}} = 2\pi\sqrt{\frac{0.098}{9.8}}\ \text{s} = \frac{\pi}{5}\ \text{s}$$

Example: How long does a pendulum have to be if it is to make one complete oscillation in one second?

Solution:
$T = 1$ s; $\ell = ?$

$$T = 2\pi\sqrt{\frac{\ell}{|g|}}$$

$$T^2 = 4\pi^2\frac{\ell}{|g|}$$

$$\ell = \frac{|g|T^2}{4\pi^2} = \frac{(9.8)(1)^2}{4\pi^2}\ \text{m} = 0.25\ \text{m}$$

Example: If a pendulum makes one complete revolution in 2 seconds on earth, how long will it take to make one complete oscillation on the moon where the gravitational acceleration is 1/6th of that on earth?

Solution: Let the subscripts, 'e' and 'm' be used for earth and moon respectively.

$T_e = 2$ s; $|g|_m = \frac{1}{6}|g|$; $T_m = ?$

$$T_e = 2\pi\sqrt{\frac{\ell}{|g|}}$$

$$T_m = 2\pi\sqrt{\frac{l}{|g|/6}}$$

$$\frac{T_m}{T_e} = \sqrt{6}$$

$$T_m = \sqrt{6}\,T_e = 2\sqrt{6}\ \text{s}$$

Example: If the length of a pendulum is doubled, by what factor will its period change?

Solution: Let the modified length and period be represented by ℓ' and T' respectively.

$$\ell' = 2\ell; \ \frac{T'}{T} = ?$$

$$\ell' = 2\ell; \ \frac{T'}{T} = ?$$

$$T = 2\pi\sqrt{\frac{\ell}{|g|}}$$

$$\frac{T'}{T} = \sqrt{2}$$

$$T' = \sqrt{2}T$$

It will be multiplied by a factor of $\sqrt{2}$.

15.5 PHYSICAL PENDULUM

A physical pendulum is an object pivoted at one of its points, displaced about the pivot by a certain angle and the let free to oscillate. Suppose an object of mass M is pivoted at one of its points and displaced by an angle θ from its equilibrium position in a counterclockwise direction. The force responsible for the rotation about the pivot is the weight $(\vec{w})$ of the object. For purposes of calculating torque, the whole weight can be assumed to act at the center of mass. Let the distance between the pivot and the center of mass of the object be d. Using a coordinate system whose origin is at the pivot, the angle formed between the positive x-axis and the position vector $(\vec{r}_{CM})$ of the center of mass is $\frac{\pi}{2}-\theta$ in a clockwise direction (that is the default angle is $-\left(\frac{\pi}{2}-\theta\right)=\theta-\frac{\pi}{2}$). Therefore the position vector of the center of mass is given as $\vec{r}_{CM} = d\cos\left(\theta-\frac{\pi}{2}\right)\hat{i} + d\sin\left(\theta-\frac{\pi}{2}\right)j = d\sin(\theta)\hat{i} - \cos(\theta)\hat{j}$. Thus the weight of the object responsible for the torque is $\vec{w} = -M|g|\hat{j}$. The torque $(\vec{\tau})$ acting on the object is $\vec{\tau} = \vec{r}_{CM} \times \vec{w} = -M|g|d\sin(\theta)\hat{k}$. For small angles $\sin(\theta)$ is approximately equal to θ. Therefore for small angular displacements, the torque acting on the object may be written as $\vec{\tau} \approx (-M|g|d)\theta\hat{k}$. But also the torque is equal to the product of the moment of inertia (I) of the object and its angular acceleration $\left(\vec{\tau} = I\alpha\hat{k} = I\frac{d^2\theta}{dt^2}\right.$ Therefore, for a physical pendulum displaced by a small angle, the differential equation $I\frac{d^2\theta}{dt^2} = -M|g|d\theta$ or

$$\frac{d^2\theta}{dt^2} + \frac{M|g|d}{I}\theta = 0$$

Applies. This is an equation of a simple harmonic motion with $\omega_0{}^2 = \frac{M|g|d}{I}$. That is, the angular frequency of a physical pendulum is given as

$$\omega_0 = \sqrt{\frac{M|g|d}{I}}$$

Example: A uniform rod of length 2m is pivoted at one of its endpoints, displaced by a small angular displacement and let free to oscillate. Its mass is 4 kg.

a) Calculate its moment of inertia about the axis of rotation.

Solution: The moment of inertia of a rod of length L and mass M about an axis passing through its center of mass perpendicularly is given as $I_{CM} = \frac{ML^2}{12}$. Its moment of inertia about one of its end points can be obtained using the parallel axis theorem: $I = I_{CM} + Md^2$ where d is the perpendicular distance between the axes.

$M = 4$ kg; $L = 2$ m; $I = ?$

$$I_{CM} = \frac{ML^2}{12} = \frac{4\times 2^2}{12}\ \text{kg m}^2 = \frac{4}{3}\ \text{kg m}^2$$

$$d = \frac{L}{2} = \frac{2}{2}\ \text{m} = 1\ \text{m}$$

$$I = I_{CM} + Md^2 = \left(\frac{4}{3} + 4\times 1^2\right)\ \text{kg m}^2 = \frac{16}{3}\ \text{kg m}^2$$

b) How long does it take to make one complete oscillation?

Solution:

$T_0 = ?$

$$\omega_0 = \sqrt{\frac{M|g|d}{I}} = \sqrt{\frac{4\times 9.8\times 1}{16/3}} = 2.7\ \text{rad/s}$$

$$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2.7}\ \text{s} = 2.3\ \text{s}$$

15.6 TORSIONAL PENDULUM

A torsional pendulum is a pendulum where a wire is twisted and let free to oscillate back and forth. In such a case the restoring torque is proportional and opposite to the angular displacement.

$$\tau = -k\theta$$

Where κ is a constant of proportionality called torsional constant. But also $\tau = I\frac{d^2\theta}{dt^2}$ where I is the moment of inertia about the axis of rotation. Therefore the following differential equation holds for a torsional pendulum.

$$\frac{d^2\theta}{dt^2} + \frac{k}{I}\theta = 0$$

But this is an equation for a simple harmonic motion with $\omega_0{}^2 = \frac{\kappa}{I}$. Therefore the angular frequency of a torsional pendulum is given as

$$\omega_0 = \sqrt{\frac{\kappa}{I}}$$

Example: A uniform disc of mass 2 kg and radius 0.1 m is hanging from a wire of torsional constant 0.1 N m attached to its center. The wire is twisted and let free to oscillate. How many oscillations will it execute for second?

Solution: The moment of inertia of a uniform disc of mass M and radius R about an axis passing through its center of mass perpendicularly is given as $I_{CM} = \frac{MR^2}{2}$.

$M = 2$ kg; $R = 0.1$ m; $\kappa = 0.1$ N m; $f_0 = ?$

$$I = I_{CM} = \frac{MR^2}{2} = \frac{2 \times 0.1^2}{2}\ \text{kg m}^2 = 0.01\ \text{kg m}^2$$

$$\omega_0 = \sqrt{\frac{\kappa}{I}} = \sqrt{\frac{0.1}{0.01}}\ \text{rad/s} = 3.16\ \text{rad/s}$$

$$f_0 = \frac{\omega_0}{2\pi} = 0.5\ \text{Hz}$$

15.7 BRIEF REVIEW OF HOMOGENOUS SECOND ORDER DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

A second order homogenous differential equation with constant coefficients has the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

Where *a*, *b* and *c* are constants. The solution of a second order differential equation is two dimensional. That is, the general solution is the linear combination of any two linearly independent (not proportional to each other) solutions of the differential equation. If $x_1(t)$ and $x_2(t)$ are any two linearly independent solutions, then the general solution of the equation can be expresses as $x(t)=c_1x_1(t)+c_2x_2(t)$ where c_1 and c_2 are arbitrary constants. c_1 and c_2 can be determined from two initial (or other) conditions. Therefore the general strategy of solving the equation is to find two linearly independent functions that satisfy the equation. Since the derivatives of e^{rt} are proportional to the function itself, it makes sense to try solutions of this form. By substituting e^{rt} (where r is a constant) directly into the equation, we can find values of *r* for which e^{rt} is a solution. Direct substitution of this function in the differential equation, results in the equation

$$ar^2+br+c=0$$

Which is called the characteristic equation of the differential equation. The function e^{rt} is a solution to the differential equation for values of r that are solutions to this quadratic equation; that is when $r=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. There are three kinds of solutions based on whether the discriminant b^2-4ac is positive, negative or positive.

Case 1: $b^2-4ac>0$

When the discriminant is positive, the quadratic equation has two distinct solutions given by $r_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$ And the general solution of the differential equation may be given as

$$x(t)=c_1e^{r_1t}+c_2e^{r_2t}$$

Where c_1 and c_2 are arbitrary constants to be determined from initial conditions.

Case 2: $b^2-4ac=0$

When the discriminant is zero, the quadratic equation has only one solution given by $r_0=-\frac{b}{2a}$. It can be shown that if the function e^{rt} is a solution of the differential equation, then the function e^{rt} is also a solution. Therefore in this case, the general solution can be taken to be the linear combination of the two independent functions e^{r_0t} and te^{r_0t}; that is

$$x(t)=(c_1+tc_2)e^{r_0t}$$

Where c_1 and c_2 are arbitrary constants to be determined from initial conditions.

Case 3: $b^2 - 4ac < 0$

In this case, there are two distinct complex solutions given by $r_1 = \dfrac{-b + i\sqrt{4ac - b^2}}{2a}$ and $r_2 = \dfrac{-b - i\sqrt{4ac - b^2}}{2a}$. With $\beta = \dfrac{b}{2a}$ and $\omega = \sqrt{4ac - b^2}$, this solutions can be written as $r_1 = -\beta + i\omega$ and $r_2 = -\beta - i\omega$. Therefore the general solution of the differential equation may be written as $x(t) = c_1 e^{(-\beta + i\omega)t} + c_2 e^{(-\beta - i\omega)t}$ where c_1 and c_2 are arbitrary constants to be determined from initial conditions. Using Euler formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, this expression can be transformed into $x(t) = e^{-\beta t}\left\{(c_1 + c_2)\cos(\omega t) + i(c_1 - c_2)\sin(\omega t)\right\}$; and with $c'_1 = c_1 + c_2$ and $c'_2 = i(c_1 - c_2)$, it can be written as $x(t) = e^{-\beta t}\left\{c'_1 \cos(\omega t) + c'_2 \sin(\omega t)\right\}$. Combining the cosine and sine into a single cosine by transforming the variables c'_1 and c'_2 into A and ϕ through the equations $c'_1 = A\cos(\phi)$ and $c_2 = A\sin\phi$, the general solution is given as

$$x(t) = Ae^{-\beta t}\cos(\omega t - \phi)$$

If $\beta = 0$, this turns out to be the solution of a simple harmonic motion as expected.

15.8 DAMPED HARMONIC MOTION

A damped harmonic motion is a motion where a particle is subjected to a resistive force proportional to the velocity of the particle in addition to the force which is proportional and opposite to the displacement. Typical resistance forces are air resistance and water resistance.

$$F_x = -cx - b\frac{dx}{dt}$$

b is the constant of proportionality between the resistive force and the velocity $v = dx/dt$. Using Newton's law $m\dfrac{d^2x}{dt^2} = -cx - b\dfrac{dx}{dt}$ or $\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{c}{m}x = 0.$ Let $\omega_o^2 = \dfrac{c}{m}$ where ω_0 is the angular frequency of the non-damped harmonic motion. Let $\dfrac{b}{m} = 2\beta$. β is called the damping constant. Then the differential equation satisfied by a particle undergoing damped harmonic motion can be written in terms of the damping constant and the angular frequency of the undamped harmonic motion as

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_o^2 x = 0$$

The characteristic equation of this differential equation which can be obtained by direct substitution of is $x(t) = e^{rt}$ is $r^2 + 2\beta + \omega_o^2 = 0$ which implies $r = -\beta \pm \sqrt{\beta^2 - \omega_o^2}$. The solution of a damped harmonic motion can be classified into three based on whether the expression $\beta^2 - \omega_0{}^2$ is positive, zero or negative.

Under Damped Oscillations: Damped oscillations occur when $\beta^2 < \omega_o^2$ and the roots are complex. With $\omega = \sqrt{\omega_o^2 - \beta^2}$, the general solution can be written as

$$x(t) = Ae^{-\beta t}\cos(\omega t - \phi)$$

ω is the angular frequency of the damped oscillation. Remember is the frequency of the harmonic motion without resistive force $(b = 0)$.

Critically Damped Motion: Critically damped motion is the transition between oscillatory and non-oscillatory damped motion. It occurs when $\beta^2 = \omega_o^2$ or when there is only one root of the characteristic equation. The general solution is given by

$$x(t) = (c_1 + c_2 t)e^{-\beta t}$$

Since $e^{-\beta t}$ decreases faster (at least eventually) than $(c_1 + c_2 t)$, the particle goes to rest without oscillation.

Over Damped Motion: Over damped motion occurs when $\beta^2 > \omega_0{}^2$ or when the characteristic equation has two real roots. The general solution is

$$x(t) = c_1 e^{\left(-\beta + \sqrt{\beta^2 - \omega_o^2}\right)t} + c_2 e^{\left(-\beta - \sqrt{\beta^2 - \omega_o^2}\right)t}$$

In this case both roots of the characteristic equation are negative which implies that both exponentials decrease with time. Since both exponentials decrease with time, the particle will eventually go to rest without any oscillations. The difference between an over damped motion and a critically damped motion is that an over damped motion goes slower to rest than a critically damped motion does.

The following diagram shows the graph of displacement versus time for the three types of a damped harmonic motion.

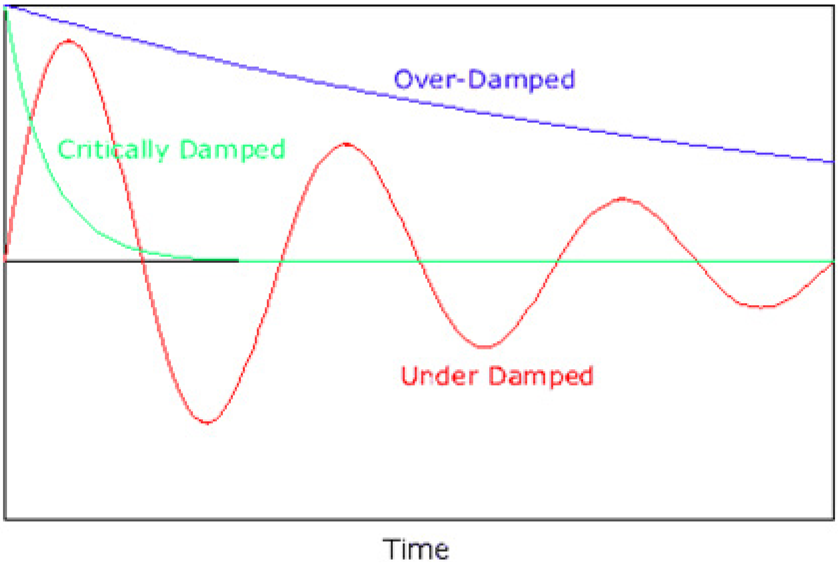

Figure 15.1

Example: A spring has a Hook's constant if 100 N/m. A particle is attached to the spring on a frictionless horizontal surface, extended and then let free. The resistive force, F_r, due to air resistance varies with the velocity of the particle according to the equation $F_r = -(2\text{ N s/m})v$ (where v is the velocity of the particle).

a) Determine the range of masses of the particle for which the motion would be under damped oscillation.

Solution: Under damped oscillation occurs when $\beta^2 - \omega_o^2 < 0$ or when $\left(\frac{b}{2m}\right)^2 < \frac{k}{m}$.
$k = 100$ N/m; $b = 2$ N s/m; $m = ?$

$$\left(\frac{b}{2m}\right)^2 < \frac{k}{m} \Rightarrow m > \frac{b^2}{4k} = \frac{2^2}{4\times 100}\text{ kg} = 0.01\text{ kg}$$

The motion of the particle will be underdamped motion when the mass of the particle is greater than 0.01 kg.

b) Determine the mass of the particle that will result in a critically damped oscillation.

Solution: Critically damped motion occurs when $\beta^2 - \omega_o^2 = 0$ or when $\left(\frac{b}{2m}\right)^2 = \frac{k}{m}$.
$m = ?$

$$\left(\frac{b}{2m}\right)^2 = \frac{k}{m} \Rightarrow m = \frac{b^2}{4k} = \frac{2^2}{4\times 100}\text{ kg} = 0.01\text{ kg}$$

The motion of the particle will be critically damped when the mass of the particle is 0.01 kg.

c) Find the range of masses of the particle for which over damped oscillation will take place.

Solution: Over damped motion occurs when $\beta^2 - \omega_o^2 > 0$ or when $\left(\frac{b}{2m}\right)^2 > \frac{k}{m}$.
$m = ?$

$$\left(\frac{b}{2m}\right)^2 > \frac{k}{m} \Rightarrow m < \frac{b^2}{4k} = \frac{2^2}{4\times 100}\ \text{kg} = 0.01\ \text{kg}$$

Over damped oscillation will take place when the mass of the particle is less than 0.01 kg.

d) If the mass of the particle is 1 kg and the spring is extended by 0.01 m and then let go

i. Calculate the damping constant.

Solution:
$m = 1$ kg; $\beta = ?$

$$\beta = \frac{b}{2m} = \frac{2}{2\times 1}\ \text{N s/m} = 1\ \text{N s/m}$$

ii. Since the mass is greater than the .01 kg, the motion should be under damped oscillation. Calculate the angular frequency of the oscillation.

Solution:
$\omega = ?$

$$\omega_o^2 = \frac{k}{m} = \frac{100}{1}\ \text{rad/s} = 100\ \text{rad/s}$$

$$\omega = \sqrt{\omega_o^2 - \beta^2} = \sqrt{100^2 - 1^2}\ \text{rad/s} = \sqrt{99}\ \text{rad/s}$$

iii. Express the displacement of the particle as a function of time.

Solution: The general solution of an under damped oscillation is given by
$x(t) = Ae^{-\beta t}\cos(\omega t - \phi)$ where A and ϕ have to be determined from initial conditions.
$x|_{t=0} = 0.01$ m; $v|_{t=0} = 0$; $x(t) = ?$

$$x(t) = Ae^{-\beta t}\cos(\omega t - \phi) = Ae^{-t}\cos\left(\sqrt{99}t - \phi\right)$$

$$x|_{t=0} = 0.01\ \text{m} \Rightarrow A\cos(\phi) = 0.01\ \text{m}$$

$$v(t) = \frac{dx}{dt} = -Ae^{-t}\cos\left(\sqrt{99}t - \phi\right) - \sqrt{99}Ae^{-t}\sin\left(\sqrt{99}t - \phi\right)$$

$$v\big|_{t=0} = 0 \Rightarrow \tan(\phi) = \frac{1}{\sqrt{99}} \text{ or } \phi = \tan^{-1}\left(\frac{1}{\sqrt{99}}\right) = 0.1 \text{ rad}$$

$$A\cos(\phi) = 0.01 \Rightarrow A = \frac{0.01}{\cos(\phi)} \text{ m} = \frac{0.01}{\cos(0.1)} \text{ m} = 0.001 \text{ m}$$

$$x(t) = .001 e^{-t}\cos\left(\sqrt{99}t - 0.1\right) \text{ m}$$

e) If a mass of 0.01 kg is attached to the spring, extended by 0.01 m and then let go, obtain an expression for the displacement of the particle as a function of time.

Solution: As shown earlier, a mass of 0.01 kg will result in a critically damped oscillation. The general solution of a critically damped oscillation is given by $x(t) = (c_1 + c_2 t)e^{-\beta t}$ where c_1 and c_2 are to be determined from initial conditions.
$m = 0.01$ kg; $x(t) = ?$

$$\beta = \frac{b}{2m} = \frac{2}{2(.01)} \text{ 1/s} = 100 \text{ 1/s}$$

$$x(t) = (c_1 + c_2 t)e^{-100t}$$

$$x\big|_{t=0} = 0.01 \text{ m} \Rightarrow c_1 = 0.01 \text{ m}$$

$$v(t) = \frac{dx}{dt} = c_2 e^{-100t} + (-100)(c_1 + c_2 t)e^{-100t}$$

$$v\big|_{t=0} = 0 \Rightarrow c_2 = 100c_1 = 100(.01) \text{ m} = 1 \text{ m}$$

$$x(t) = (0.01 + t)e^{-100t} \text{ m}$$

f) If a mass of 0.005 kg is attached to the spring, extended by 0.01 m and then let go, find an expression for the displacement of the particle as a function of time.

Solution: As shown earlier, since 0.005 kg < .01 kg the motion will be an over damped oscillation. The general solution of an over damped motion is given by $x(t) = c_1 e^{\left(-\beta + \sqrt{\beta^2 - \omega_o^2}\right)t} + c_2 e^{\left(-\beta - \sqrt{\beta^2 - \omega_o^2}\right)t}$ where c_1 & c_2 are to be determined from initial conditions.
$m = 0.005$ kg; $x(t) = ?$

$$\beta = \frac{b}{2m} = \frac{2}{2(.005)} \text{ 1/s} = 200 \text{ 1/s}$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{.005}} \text{ rad/s} = 100\sqrt{2} \text{ rad/s}$$

$$\sqrt{\beta^2-\omega_o^2}=100\sqrt{2}\ 1/\mathrm{s}$$

$$\beta+\sqrt{\beta^2-\omega_o^2}=100\left(-2+\sqrt{2}\right)\ 1/\mathrm{s}$$

$$\beta-\sqrt{\beta^2-\omega_o^2}=100\left(-2-\sqrt{2}\right)\ 1/\mathrm{s}$$

$$x(t)=c_1e^{\left(-2+\sqrt{2}\right)100t}+c_2e^{\left(-2-\sqrt{2}\right)100t}$$

$$x\big|_{t=0}=0.01\text{ m}\Rightarrow c_1+c_2=0.01\text{ m}$$

$$v(t)=\frac{dx}{dt}=\left(-2+\sqrt{2}\right)c_1e^{\left(-2+\sqrt{2}\right)100t}+\left(-2-\sqrt{2}\right)c_2e^{\left(-2-\sqrt{2}\right)100t}$$

$$v\big|_{t=0}=0\Rightarrow\left(-2+\sqrt{2}\right)c_1=\left(2+\sqrt{2}\right)c_2$$

$$c_1+c_2=0.01\text{ m and }\left(-2+\sqrt{2}\right)c_1=\left(2+\sqrt{2}\right)c_2\Rightarrow c_1=0.012\text{ m and }c_2=-.002\text{ m}$$

$$x(t)=(.012\text{ m})e^{\left(-2+\sqrt{2}\right)100t}-(.002\text{ m})e^{\left(-2-\sqrt{2}\right)100t}$$

Practice Quiz 15.2

Choose the best answer

1. The motion of a pendulum is approximately harmonic if
   A) The pendulum is displaced by a small angle.
   B) The pendulum is very short.
   C) The pendulum is very long.
   D) The pendulum is displaced by a large angle (close to 90 degree)
   E) The mass of the pendulum is small.

2. How long will a pendulum of length *7* m take to make one complete oscillation?
   A) *7.518* s
   B) *5.31* s
   C) *4.299* s
   D) *6.228* s
   E) *3.642* s

3. A pendulum makes *7* cycles per second on earth. How many cycles does it execute per second on the moon where gravitational acceleration is $(1/6)^{th}$ of that on earth?
   A) *4.814* Hz
   B) *0.86* Hz
   C) *2.858* Hz
   D) *0.465* Hz
   E) *3.649* Hz

4. A physical pendulum consists of a uniform rod of mass *8.1* kg and length *0.7* m pivoted at one of its ends. The rod is displaced by a small angle and let free to oscillate. How long does it take to make one complete revolution. ($I_{CM} = ML^2/12$)
   A) *0.971* S
   B) *1.371* S
   C) *2.27* S
   D) *0.535* S
   E) *2.008* S

5. A uniform sphere of mass *6.4* kg and radius *0.056* m is suspended from its surface by a wire of torsional constant *64* Nm. The wire is twisted and let free to oscillate. How long does it take to make one complete revolution. ($I_{CM} = 2MR^2/5$)
   A) *10.42e-2* s
   B) *9.343e-2* s
   C) *12.876e-2* s
   D) *5.721e-2* s
   E) *7.037e-2* s

6. The equation of motion of a certain particle subjected to a force proportional and opposite to its displacement and a resistive force proportional and opposite to its velocity is $d^2x/dt^2 + 3.6dx/dt + 36x = 0$ Calculate the natural frequency of the motion.
   A) *0.189* Hz
   B) *1.417* Hz
   C) *1.77* Hz
   D) *0.752* Hz
   E) *0.955* Hz

7. A particle is moving under the influence of a restoring and resistive forces given by $F_{restoring} = -40x$ and $F_{resistive} = -3.6v$ respectively. (x and v represent displacement and velocity of the particle respectively). What are the possible values for the mass (m) of the particle if the motion is to be an under damped oscillation.
   A) $m < 8.1e\text{-}2$ kg
   B) $m > 15.122e\text{-}2$ kg
   C) $m = 8.1e\text{-}2$ kg
   D) $m < 15.122e\text{-}2$ kg
   E) $m > 8.1e\text{-}2$ kg

8. The equation of motion of a particle of mass 8.36 kg subjected to a restoring force proportional to its displacement and a resistive force proportional to its velocity is $d^2x/dt^2 + 38.7dx/dt + 51.9x = 0$ The resistive force varies with velocity as
   A) $F_{resistive} = -419.38v$
   B) $F_{resistive} = -467.941v$
   C) $F_{resistive} = -217.657v$
   D) $F_{resistive} = -611.738v$
   E) $F_{resistive} = -323.532v$

9. The equation of motion of a certain particle subjected to a restoring force proportional and to its displacement and a resistive force proportional to its velocity is $d^2x/dt^2 + 8.2dx/dt + 16.81x = 0$ The general solution for the displacement as a function of time in terms of arbitrary constants A and B is.
   A) $(A + Bt)e^{-4.92t}$
   B) $A\ e\text{-}4.1t + B\ e^{4.1t}$
   C) $A\ e^{-4.1t} \cos(16.81t - B)$
   D) $(A + Bt)e^{-4.1t}$
   E) $A\ e^{-4.92t} + B\ e^{6.07t}$

10. A particle of mass 4.61 kg is moving under the influence of a restoring and resistive forces given by $F_1 = -40x$ and $F_2 = -3.23v$ respectively. (x and v represent displacement and velocity of the particle respectively). If the particle was released from a displacement of 1.8 m, give a formula for its displacement as a function of time
   A) $2.077\ e^{-0.35t} \cos(2.925t - 0.078)$
   B) $1.813\ e^{-0.35t} \cos(2.925t - 0.196)$
   C) $0.775\ e^{-0.35t} \cos(2.925t - 0.196)$
   D) $0.775\ e^{-0.35t} \cos(2.925t - 0.119)$
   E) $1.813\ e^{-0.35t} \cos(2.925t - 0.119)$

ANSWERS TO PRACTICE QUIZZES

Practice Quiz 1.1

1. A 2. C 3. B 4. C 5. B 6. C 7. C 8. B 9. C 10. A 11. C 12. B

Practice Quiz 1.2

1. D 2. A 3. A 4. A 5. B 6. E 7. C 8. D 9. C 10. B 11. E 12. C 13. D

Practice Quiz 2.1

1. E 2. B 3. C 4. E 5. D 6. B 7. C 8. A 9. E 10. A

Practice Quiz 2.2

1. D 2. B 3. C 4. C 5. A 6. A 7. A 8. B 9. D 10. C 11. D

Practice Quiz 3.1

1. C 2. A 3. D 4. D 5. A 6. A 7. E 8. D 9. E 10. C 11. B 12. E 13. C 14. D 15. B

Practice Quiz 3.2

1. E 2. E 3. C 4. A 5. B 6. A 7. A 8. A 9. B 10. D 11. C 12. E

Practice Quiz 4.1

1. A 2. D 3. B 4. D 5. D 6. C 7. E 8. C 9. A 10. E 11. B

Practice Quiz 4.2

1. A 2. B 3. A 4. B 5. E 6. E 7. D 8. B 9. E 10. B 11. B

Practice Quiz 5.1

1. D 2. C 3. E 4. A 5. B 6. D 7. C 8. D 9. D 10. C 11. A

Practice Quiz 5.2

1. C 2. C 3. D 4. A 5. E 6. D 7. C 8. C 9. D 10. C 11. C

Practice Quiz 6.1

1. C 2. A 3. C 4. E 5. C 6. B 7. D 8. A 9. D 10. B 11. D

Practice Quiz 6.2

1. B 2. E 3. B 4. A 5. B 6. C 7. C 8. C 9. A 10. B

Practice Quiz 7.1

1. C 2. A 3. C 4. A 5. A 6. E 7. E 8. A 9. D 10. A

Practice Quiz 7.2

1. A 2. C 3. B 4. B 5. D 6. A 7. A 8. E 9. A 10. E 11. B 12. B

Practice Quiz 8.1

1. E 2. A 3. A 4. C 5. C 6. C 7. D 8. B 9. D 10. D 11. B 12. D

Practice Quiz 8.2

1. B 2. C 3. E 4. E 5. E 6. A 7. C 8. C 9. B 10. B 11. B 12. C

Practice Quiz 9.1

1. A 2. D 3. C 4. E 5. C 6. E 7. A 8. A 9. E 10. E

Practice Quiz 9.2

1. D 2. B 3. B 4. D 5. A 6. B 7. B 8. E 9. E 10. B

Practice Quiz 10.1

1. C 2. A 3. C 4. E 5. B 6. B 7. A 8. D 9. A 10. D 11. C

Practice Quiz 10.2

1. B 2. D 3. D 4. B 5. C 6. D 7. E 8. B 9. A 10. E

Practice Quiz 11.1

1. E 2. E 3. D 4. A 5. E 6. E 7. C 8. D 9. A 10. C 11. C

Practice Quiz 11.2

1. E 2. C 3. C 4. E 5. D 6. B 7. B 8. C 9. A 10. B 11. B 12. C

Practice Quiz 12.1

1. A 2. D 3. D 4. B 5. E 6. B 7. B 8. D 9. B 10. C

Practice Quiz 12.2

1. A 2. C 3. D 4. D 5. A 6. E 7. C 8. D

Practice Quiz 13.1

1. A 2. B 3. B 4. D 5. B 6. B 7. B 8. B 9. C 10. B

Practice Quiz 13.2

1. E 2. E 3. B 4. E 5. D 6. A 7. C 8. E 9. B 10. B

Practice Quiz 14.1

1. D 2. A 3. C 4. E 5. A 6. A 7. A 8. C 9. B 10. E 11. C

Practice Quiz 14.2

1. B 2. E 3. B 4. C 5. D 6. B 7. B 8. B 9. A 10. C

Practice Quiz 15.1

1. E 2. B 3. B 4. D 5. D 6. E 7. B 8. C 9. A 10. D 11. E

Practice Quiz 15.2

1. A 2. B 3. C 4. B 5. E 6. E 7. E 8. E 9. D 10. E